Change of Basis (6.5)

$$
\begin{tikzcd}
 & v \in V \arrow[dl] \arrow[dr] & \\
{[v]_{B_1}} & & {[v]_{B_2}}
\end{tikzcd}
$$

How are coordinates of a vector in different bases of V related?

$$
\begin{tikzcd}
V \arrow[r, "id"] \arrow[d, leftrightarrow, "C_{(B_1)}"'] & V \arrow[d, leftrightarrow, "C_{(B_2)}"'] \\
{[v]_{B_1}} \arrow[r, "\mu_P"] & {[v]_{B_2}}
\end{tikzcd}
$$

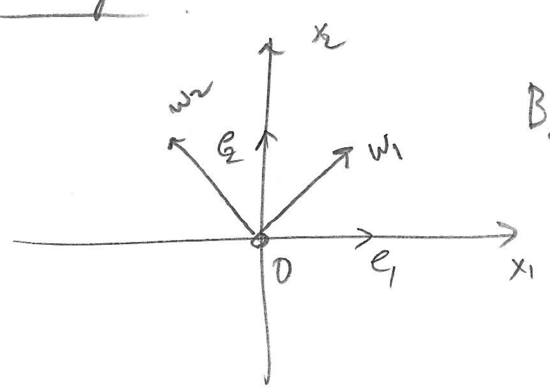
$B_1 = \{v_1 \dots v_n\}$

$B_2 = \{w_1 \dots w_n\}$

$$P = \left[[v_1]_{B_2} \; [v_2]_{B_2} \; \dots \; [v_n]_{B_2} \right]$$

Example.

New basis is turned by 45° counter clockwise



$$B_2 \begin{cases} w_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\ w_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{cases}$$

$$[e_1]_{B_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

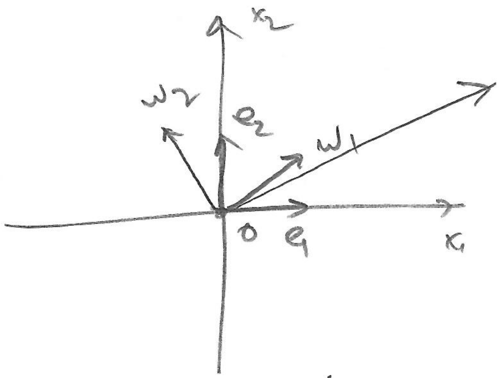
$$B_1 = \{e_1, e_2\}$$

$$[e_2]_{B_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

If $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in basis B_1, then

$$[v]_{B_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$



$$3e_1 + 2e_2 = \frac{5}{\sqrt{2}} w_1 - \frac{1}{\sqrt{2}} w_2 .$$

How to find coordinates in the new basis?
(if it is not the standard one)

Here $w_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; $w_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

in the basis B_1.

The matrix $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

lists coordinates of the basis B_2 in the basis B_1
(the standard one)

R performs the change of coordinates from new basis to the standard basis.

So R is the inverse of P!
(i.e. P is the inverse of R)

$$R^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} = \frac{1}{\left(\frac{1}{2}+\frac{1}{2}\right)} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\left(\text{using} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

<u>Example</u>: <u>#1</u> Find the change-of-basis matrix from

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \text{ to } B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Use $B_0 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

$$\begin{tikzcd}
v \in V \arrow[r, "id"] \arrow[d, leftrightarrow] & v \in V \arrow[r, "id"] \arrow[d, leftrightarrow] & v \in V \arrow[d, leftrightarrow] \\
{[v]_{B_1}} \arrow[r, "P"] & {[v]_{B_0}} \arrow[r, "Q"] & {[v]_{B_2}}
\end{tikzcd}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad Q^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{1-4} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -1 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -1 \end{bmatrix}.$$

Example: $V = \mathbb{P}_2$;

$(\sim \#2, 6.5)$ $B_1 = \{1-x, x^2, 1+x\}$ $B_2 = \{x, 1+x^2, 1-x^2\}$.

Find the change of base matrix from B_1 to B_2

$$P = \left[[1-x]_{B_2}, [x^2]_{B_2}, [1+x]_{B_2} \right].$$

$$1 - x = \frac{1}{2}\left((1+x^2) + (1-x^2)\right) - x$$

so $$[1-x]_{B_2} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$x^2 = \frac{1}{2}\left((1+x^2) - (1-x^2)\right), \text{ so } [x^2]_{B_2} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Similarly,

$$[1+x]_{B_2} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

So, $$P = \begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

[Another way: $B_0 = \{1, x, x^2\}$;

$$\begin{tikzcd}
p(x) \arrow[r, "id"] \arrow[d, "(B_1)"'] & p(x) \arrow[r, "id"] \arrow[d, leftrightarrow, "(B_0)"'] & p(x) \arrow[d, leftrightarrow] \\
{[p(x)]_{B_1}} \arrow[r, "P_1"] & {[p(x)]_{B_0}} \arrow[r, "P_2"] & {[p(x)]_{B_2}}
\end{tikzcd}$$

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q_2 = P_2^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

To find $P_2 P_1$ reduce $[Q_2 \mid P_1]$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \quad Q_2^{-1} P_1 = P_2 P_1 = \begin{bmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Change of basis for a linear Transformation.

Suppose $T: V \to V$ is such that

$$[T(v)]_B = A\,[v]_B .$$

If we use a different basis B',
how do we determine the matrix A' s.t.

$$[T(v)]_{B'} = A'\,[v]_{B'} \;?$$

If P is the change-of base matrix from B' to B, then P^{-1} is the change of base from B to B', so

$$[T(v)]_{B'} = P^{-1}[T(v)]_B = P^{-1}\left(A\,[v]_B\right)$$
$$= P^{-1}\left(A\left(P[v]_{B'}\right)\right)$$
$$= \left(P^{-1}AP\right)[v]_{B'}$$

This gives the formula

$$\boxed{A' = P^{-1}AP.}$$

(P change of base from B' to B!)

Example $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

(cf. #4 in Sect. 6.5)

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3\}$$

The matrix of T in basis B is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Find the matrix of T in the standard basis.

$$B_0 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. = \{e_1, e_2, e_3\}$$

According to the formula,

$$A_0 = P^{-1} A P$$

where P is the change of base matrix from B_0 to B.

Easier to find $Q = P^{-1}$: from B to B_0:

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

for P use $\quad e_1 = v_1 \Rightarrow [e_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$e_2 = v_2 - v_1 \Rightarrow [e_2]_B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$e_3 = v_3 - v_2 + 2v_1 \Rightarrow [e_3]_B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

So $\quad P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Note: To find P from Q we could also reduce

$$[Q \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

To find A_0, work the matrix multiplications:

$$AP = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

$$QAP = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

(b) let $B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Find the matrix P of changing coordinates from B' to B.

$$P = \left[\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_B, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_B \right]$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Compute the matrix of T in B':

$$P^{-1}AP$$

$$AP = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 12 \\ 1 & -1 & -3 \\ -1 & 2 & 7 \end{bmatrix}$$

To find $P^{-1}AP$, reduce $[P \,\vdots\, AP]$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 0 & 4 & 12 \\ 1 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 & 2 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 2 & 0 & 4 & 12 \\ 0 & 0 & 1 & -1 & 2 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 & 2 & 7 \end{array}\right]$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & -3 \\ 2 & 0 & -2 \\ -1 & 2 & 7 \end{bmatrix}.$$

(d) Using A, compute $T(e_1), T(e_2), T(e_3)$

Computed: $[e_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[e_2]_B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$[T(e_1)]_B = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$[T(e_2)]_B = A\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow T(e_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$[T(e_3)]_B = A\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow T(e_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Example: The matrix of T in the standard (B_0) basis is

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

Find the matrix of T in the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{- from } B \text{ to } B_0$$

$$AP = \begin{bmatrix} 1 & -2 & 6 \\ 0 & 2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$[P \mid AP] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & -2 & 6 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 & 2 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{- diagonal matrix!}$$

Exercises 6.5: 2, 4, 5