Matrices of Compositions and Inverses (6.4)

Recall:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow, "\text{coord. } B"'] & W \arrow[d, leftrightarrow, "\text{coord. } B'"] \\ \mathbb{R}^n \arrow[r, "\mu_A"] & \mathbb{R}^m \end{tikzcd}$$

Where μ_A is mult. by matrix A,

$$A = \left[[T(v_1)]_{B'} ; \dots [T(v_n)]_{B'} \right]$$

columns of coordinates arranged into columns of the matrix A.

Composition:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow, "c\,(B_0)"'] & W \arrow[r, "U"] \arrow[d, leftrightarrow, "c\,(B')"'] & Z \arrow[d, leftrightarrow, "c\,(B'')"'] \\ \mathbb{R}^n \arrow[r, "\mu_A"] & \mathbb{R}^m \arrow[r, "\mu_B"] & \mathbb{R}^\ell \end{tikzcd}$$

Suppose V has basis $B_0 = \{v_1 .. v_n\}$
W has basis $B' = \{w_1 .. w_m\}$
and Z has basis $B' = \{z_1 ... z_\ell\}$

According to the diagram, to obtain coordinates of $U(T(v))$ in the basis B'' we need to multiply the column of coordinates of $T(v)$ on the basis B' by the matrix B.

i.e.

$$[U(T(v))]_{B''} = B\,[T(v)]_{B'}$$
$$= B\left(A\,[v]_{B_0}\right)$$
$$= (BA)\,[v]_{B_0}$$

using associative property of matrix times matrix.

[Theorem 6.13] The matrix corresponding to composition of linear transformations $U \circ T$ is the product of the matrices BA.

Example: $T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad U: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_3 \\ 3x_1 \end{bmatrix}$$

$$U\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_2 \\ y_1 \\ 0 \end{bmatrix}$$

In the standard basis:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U\left(T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)\right) = \begin{bmatrix} -3x_1 \\ 2x_1 - x_3 \\ 0 \end{bmatrix}$$

So the matrix of $U \circ T$ is $\begin{bmatrix} -3 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

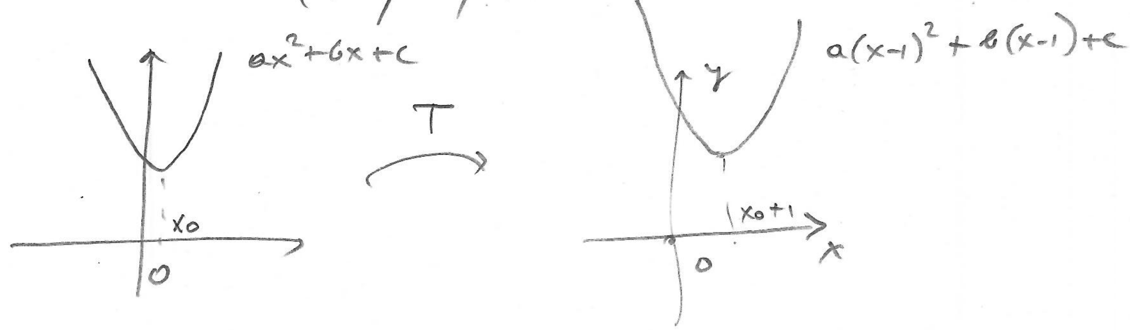
Check:

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

Example (p. 234)

$T : \mathbb{P}_2 \to \mathbb{P}_2 \quad T(ax^2+bx+c) = a(x-1)^2 + b(x-1)+c$

(T performs "right shift by +1")



$D : \mathbb{P}_2 \to \mathbb{P}_2 \quad D(ax^2+bx+c) = 2ax+b$

(differentiation)

In the standard bases

$\{1, x, x^2\}$, and $\{1, x, x^2\}$

The matrix of T is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of D is $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

The composition $D \circ T$ is

$$D \circ T(ax^2 + bx + c) = 2a(x-1) + b$$

so the matrix of $D \circ T$ is

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Compute the product:

$$BA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

<u>Inverses</u>

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, leftrightarrow, "C_{(B_0)}"'] & W \arrow[r, "T^{-1}"] \arrow[d, leftrightarrow, "C_{(B_1)}"'] & V \arrow[d, leftrightarrow, "C_{(B_0)}"'] \\
\mathbb{R}^n \arrow[r, "\mu_A"'] & \mathbb{R}^n \arrow[r, "\mu_B"'] & \mathbb{R}^n
\end{tikzcd}$$

Since $T^{-1} \circ T = id_V$

$$[T^{-1} \circ T(v)]_{B_0} = [v]_{B_0}$$

So the matrix of the composition is the identity matrix

$$I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 1 \end{bmatrix}$$ of size $n \times n$.

Thus for any $v \in V$

$$BA[v]_{B_0} = [v]_{B_0}$$

$$BA = I$$

and similarly (consider $T \circ T^{-1}$)

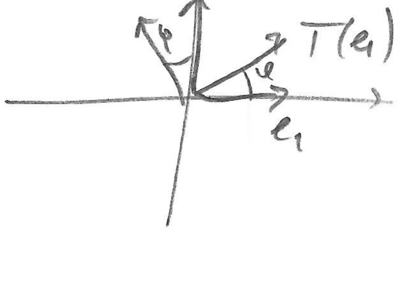
$$AB = I$$

According to the definition of matrix inverse, the matrix B of transformation T^{-1} must be the inverse of the matrix A.

[Theorem 6.14] : If $T: V \to W$ is invertible then the matrix of T (denoted by A) is invertible, and the inverse of A is the matrix of T^{-1}.

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$
- rotation through angle φ (in radians)

Use standard basis of $\mathbb{R}^2$:
$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$



$$T(e_1) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

The inverse of T is the rotation through $-\varphi$

(i.e. clockwise, rather than counterclockwise)

$$B = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

The matrix B is the inverse of A:

$$BA = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi(-\sin\varphi) + \sin\varphi\cos\varphi \\ (-\sin\varphi)\cos\varphi + \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And also

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(check!)}$$

Example problem #2 (6.4)

$$L : \mathbb{P}_2 \to \mathbb{P}_2 \qquad L\big(p(x)\big) = 2p(x+1)$$

(shift to the left by 1 and scale by a factor of 2.)

$B = \{1, x, x^2\}$.

(a) Find the matrix of L on B, B. (used both for the domain and the range.)

$L(1) = 2$; $L(x) = 2(x+1) = 2x+2$

$L(x^2) = 2(x+1)^2 = 2x^2 + 4x + 2$

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) The inverse of L is "scale down by factor of 2, then shift to the right by 1"

$$L^{-1}\big(p(x)\big) = \frac{1}{2}\, p(x-1)$$

Indeed $L^{-1}\big(L(p(x))\big) = L^{-1}\big(2\, p(x+1)\big)$

$$= \frac{1}{2} \cdot 2\, p\big((x+1)-1\big)$$

$$= p(x)$$

and $L\big(L^{-1}(p(x))\big) = L\big(\frac{1}{2}\, p(x-1)\big)$

$$= 2\Big(\frac{1}{2}\, p\big((x-1)+1\big)\Big) = p(x).$$

(c) The matrix of L^{-1} is :

$$L(1) = \frac{1}{2} \qquad L(x) = \frac{1}{2}(x-1) = \frac{1}{2}x - \frac{1}{2}$$

$$L(x^2) = \frac{1}{2}(x-1)^2 = \frac{1}{2}x^2 - x + \frac{1}{2}.$$

$$B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(d) check that B and A are inverses;

$$BA = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Note</u>: Since both A and B are square,

$BA = I$, A has left inverse $\Rightarrow$

A has right inverse as well [illegible]

Exercises 6.4 : 1, 5, 6