Matrix of a linear transformation (6-3)

$f: V \to W$, linear, where

$V = \mathbb{R}^n$; $W = \mathbb{R}^m$.

Goal: Represent f by an $m \times n$ matrix.

Example: (1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ -3x \end{bmatrix}$ – linear $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

(2) Suppose T: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$; $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x\begin{bmatrix} 9 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So the columns of the matrix of T are the values of T on the vectors of the standard basis of $\mathbb{R}^2$.

This is true in general:

[Theorem 6.10] Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear

Let $\{e_1, e_2 \dots e_n\}$ – standard basis in $\mathbb{R}^n$

i.e. $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith position.

Consider the matrix A for which the columns are $T(e_i)$, $i = 1 \dots n$.

Then $T(v) = Av$.

<u>Proof</u>: $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} v_i e_i$

$T(v) = T\left(\sum_{i=1}^{n} v_i e_i\right) = \sum_{i=1}^{n} v_i T(e_i) = Av$

For more general vector spaces:

$T: V \to W$

Represent $v \in V$ and $T(v) \in W$ by coordinates:

$[v]_B$, $[T(v)]_{B'}$.

Then $[T(v)]_{B'} = A[v]_B$, where

A is the matrix with the columns

$[T(v_1)]_{B'}, \dots [T(v_n)]_{B'}$

Indeed if $[v]_B = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then

$T(v) = v_1 T(v_1) + \dots + v_n T(v_n)$

$[T(v)]_{B'} = \left[[T(v_1)]_{B'} ; \dots [T(v_n)]_{B'} \right] [v]_B$.

Example: $T: \mathbb{P}_2 \to \mathbb{P}_2$

given by $T(a + bx + cx^2) = a + b(x-1) + c(x-1)^2$

Consider $B = \{1, x, x^2\}$; $B' = \{1, x, x^2\}$

Then

$$[p(x)]_B = [a + bx + cx^2]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[T(p(x))]_{B'} = [a + b(x-1) + c(x-1)^2]_{B'}$$

$$= [a - b + bx + c - 2cx + cx^2]_{B'}$$

$$= [(a - b + c) + (bx - 2cx) + cx^2]_{B'}$$

$$= \begin{pmatrix} a - b + c \\ b - 2c \\ c \end{pmatrix}$$

So

$$[T(p(x))]_{B'} = \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The columns of the matrix A correspond to what's multiplied by a, b, c, respectively.

So they are obtained from applying T to 1, x, and x^2 respectively, and expressing in the basis B':

$$[T(1)]_{B'} = [1]_{B'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x)]_{B'} = (x-1)_{B'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(x^2)]_{B'} = [(x-1)^2]_{B'} = [x^2 - 2x + 1]_{B'} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Suppose we use a different basis for W:

$$B' = \{1, (x-1), (x-1)^2\} \quad \text{while} \quad B = \{1, x, x^2\}.$$

Then $$[p(x)]_B = [a + bx + cx^2]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and $$[T(p(x))]_{B'} = [a + b(x-1) + c(x-1)^2]_{B'} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So $$[T(p(x))]_{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and the matrix of T in the bases B, B' becomes a 3×3 <u>identity</u> matrix.

<u>To summarize:</u> A linear transformation $T : V \to W$ can be represented by a matrix $A \in M(m,n)$, which is determined by a choice of bases B for V and B' for W.

Depending on the choice of bases, the matrix could be simpler, or more complicated.

<u>Diagram:</u>

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, leftrightarrow, "\text{coordinates in } B"'] & W \arrow[d, leftrightarrow, "\text{coordinates in } B'"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m
\end{tikzcd}$$

<u>Exercises</u> 6.3: 1, 3, 8, 10, 11.