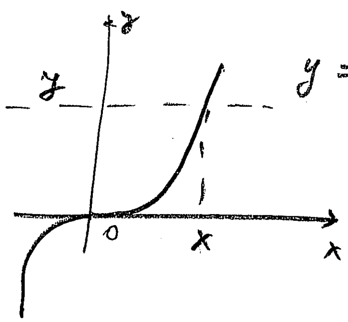
Compositions and Inverses. (6.2)

Idea of inverse function:

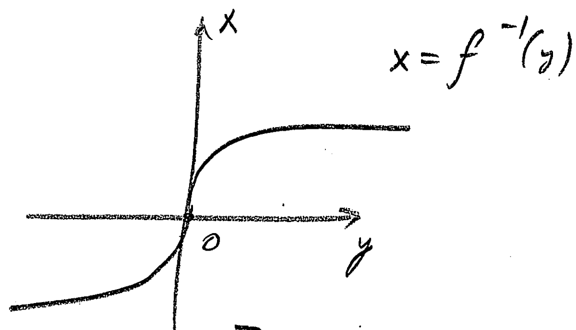
$$f(x) = y \quad \text{— equation to be solved for } x$$

If for any y in the range of the function there is one and only one x, this defines a function

$$x = f^{-1}(y).$$



Direct function



Inverse function

If f has an inverse, f^{-1}, we say f is <u>invertible</u>.

In order to be invertible a function needs to satisfy the "horizontal line test":

"for any y there is only one x"

or "$(f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$"

which is logically equivalent to

"$(x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))$".

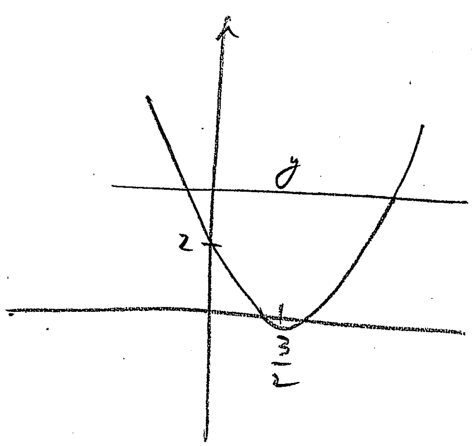
A better name for the "horizontal line test" is "one-to-one".

$$\left(f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \right).$$

We also say a function $f: X \to Y$ is <u>onto</u> Y if for any $y \in Y$ there is an $x \in X$ such that $f(x) = y$.

<u>Example</u>. $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2 - 3x + 2$$



f is <u>not</u> one to one:

$$f\left(\tfrac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 2 = -0.25$$

for any $y > -0.5$

there are two x-values

$$x = -\frac{3}{2} \pm \sqrt{y + 0.25}$$

f is <u>not</u> onto $\mathbb{R}$;

if $y < -0.5$ then there is no $x \in \mathbb{R}$ such that $f(x) = y$.

$f: \mathbb{R} \to [-0.25, \infty)$ <u>is</u> onto.

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$ (p. 220)

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ -3x \end{bmatrix}$$

Show that T is 1-to-1 and onto $\mathbb{R}^2$.

1-to-1:
$$\begin{bmatrix} 2x_1+y_1 \\ -3x_1 \end{bmatrix} = \begin{bmatrix} 2x_2+y_2 \\ -3x_2 \end{bmatrix}$$

$$-3x_1 = -3x_2 \Rightarrow x_1 = x_2$$

$$2x_1+y_1 = 2x_2+y_2 \Rightarrow y_1 = y_2$$

onto:
$$\begin{bmatrix} 2x+y \\ -3x \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & a \\ -3 & 0 & b \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{a}{2} \\ 0 & \frac{3}{2} & \frac{3a}{2}+b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{a}{2} \\ 0 & 1 & a+\frac{2}{3}b \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{b}{3} \\ 0 & 1 & a+\frac{2}{3}b \end{array}\right]$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is determined (uniquely) for any $\begin{bmatrix} a \\ b \end{bmatrix}$.

Theorem 6.6 $f: X \to Y$ is invertible if and only if f is 1-to-1 and onto Y.

[No proof.]

Theorem [6.7–6.8] Linear maps: Composition and Inverses.

(1) $U \xrightarrow{S} V \xrightarrow{T} W$ (U, V, W – vector spaces.)

S, T – linear $\Rightarrow$ $T \circ S$ is linear.

(2) If $T: V \to W$ is invertible then T^{-1} is linear

Proof: (1) $T(S(x+y)) = T(S(x) + S(y))$
$= T(S(x)) + T(S(y))$

$T(S(rx)) = T(rS(x)) = rT(S(x))$.

(2) $T^{-1}(y_1 + y_2) = T^{-1}(T(x_1) + T(x_2))$
$= T^{-1}(T(x_1 + x_2)) = x_1 + x_2 = T^{-1}(y_1) + T^{-1}(y_2)$

$T^{-1}(ry) = T^{-1}(rT(x)) =$
$= T^{-1}(T(rx)) = rx = rT^{-1}(y)$.

Exercises 6.2: 1ce, 2ce, 8ce