Linear functions (6.1)

V, W - vector spaces

Def: $T: V \to W$ is linear if for any $v, w \in V$, $r \in \mathbb{R}$

$$T(v+w) = T(v) + T(w) \quad \text{(additive)}$$
$$T(rv) = rT(v). \quad \text{(homogeneous)}$$

Other terms: Linear transformation
Linear map
Linear operator (for the case $V=W$)

Example: $V = \mathbb{R}^n$ (use vector-columns for convenience.)

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A \in M(m,n) \quad ; \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Then $\mu_A(x) = Ax$

is a linear function, with values in $\mathbb{R}^m$ (arranged as column)

$$A(x+y) = Ax + Ay \quad \checkmark$$
$$A(rx) = rAx \quad \checkmark$$

Basic Properties of Linear Maps

[Theorem 6.2]

(a) $T(0_V) = 0_W$

Indeed for any $x \in V$

$$T(rx) = rT(x),$$

for $r=0$ $T(0_V) = 0_W$.

(b) $T(-v) = -T(v)$

Use: $-v = 0-v$, $T(0)=0$

(c) $$T(r_1 v_1 + \dots + r_n v_n) = r_1 T(v_1) + \dots + r_n T(v_n)$$

If $n=2$:

$$T(r_1 v_1 + r_2 v_2) = T(r_1 v_1) + T(r_2 v_2)$$
$$= r_1 T(v_1) + r_2 T(v_2).$$

General n: induction.

<u>Theorem 6.3</u> $S = \{v_1, \dots v_n\}$ spans V

$$\left.\begin{matrix} T: V \to W \\ \tilde{T}: V \to W \end{matrix}\right\} \text{linear},$$

$T(v_i) = \tilde{T}(v_i)$, $i=1..n$

Then for any v, $T(v) = \tilde{T}(v)$.

Proof:

$$T(v) = T\left(\sum r_i v_i\right)$$
$$= \sum r_i T(v_i)$$
$$= \sum r_i \tilde{T}(v_i)$$
$$= \tilde{T}\left(\sum r_i v_i\right)$$
$$= \tilde{T}(v)$$

Examples: #1: $P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

Find A s.t. $P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$P\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$P\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$P\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

#2: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 5x+3$
is not linear

$$f(2x) = 10x+3 \neq 2(5x+3)$$
$$= 10x+6$$

Also $f(0) = 3 \neq 0$

Exercises: 6.1: 8, 15, 17, 18, 21