MATH 262

Matrix Inverses (5.2)

A system of lin. equs.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

can be reformulated using the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$ and the vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

as a vector equation

$$Ax = b.$$

(matrix-times-a-vector operation is used to produce the LHS)

We would like to solve it as

$$\text{"} x = \frac{b}{A} \text{"}$$

Indeed, if we know a matrix B such that $BA = I$ then

$$B(AX) = Bb$$
$$(BA)X = Bb$$
$$IX = Bb$$
$$X = Bb.$$

So far, this shows that "IF $Ax = b$ has a solution THEN $x = Bb$".

But is it really true that 'x' obtained in this way satisfies $Ax = b$?

Plug it back in:

$$A(Bb) = (AB)b = Ib = b$$

<u>Provided</u> $AB = I$

(So B is also a right inverse of A.)

<u>Def</u>: If $A \in M(m,n)$ then

$B \in M(n,m)$ is a <u>left</u> inverse provided $BA = I_n$

$B \in M(n,m)$ is a <u>right</u> inverse if $AB = I_m$

B is an <u>inverse</u> if $BA = I_n$, $AB = I_m$.

(i.e. both left and right.)

If A has an inverse, we say "A is invertible".

Example: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a right inverse

$\tilde{B} = \begin{bmatrix} 0 & 0 \\ -1 & 3 \\ 1 & -2 \end{bmatrix}$ is also a right inverse

$(AB = I_2, \ A\tilde{B} = I_2)$.

For $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is a left inverse

$\tilde{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$ is also a left inverse.

Proposition: If $AB = I_m$ and $CA = I_n$
Then $B = C$.

Proof: $B = I_n B = (CA)B = C(AB) = CI_m = C$

Thus, if a matrix is invertible then the left and the right inverse are the same.

Corollary: [Theorem 5.7] If A is invertible then the inverse matrix B is unique.

How to find a matrix inverse?

Solve $AB = I$ (for right inverse)

For simplicity, let $A, B \in M(3,3)$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of B satisfy $(B = [C_1, C_2, C_3])$

$$A C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad A C_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To solve these equations simultaneously, form an augmented matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

and reduce it:

Example:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_3 \leftrightarrow R_2 \\ -R_2 \to R_2 \\ -4R_2 + R_3 \to R_3 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & -1 & -7 & 1 & 4 \end{array}\right]$$

$$\begin{matrix} -R_3 \to R_3 \\ -R_3 + R_2 \to R_2 \\ -R_3 + R_1 \to R_1 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -6 & 1 & 4 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & 7 & -1 & 4 \end{array}\right]$$

$$-R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & 7 & -1 & 4 \end{array}\right]$$

So $B = \begin{bmatrix} -1 & 0 & 1 \\ -5 & 1 & 3 \\ 7 & -1 & 4 \end{bmatrix}$ satisfies $AB = I$.

In this case also $BA = I$, so A is <u>invertible</u>.

What if we take a matrix A that is not square:

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

reduces to

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

If there are leading ones in every row,

then we will be able to find a solution, not unique.

For instance, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, or $\tilde{B} = \begin{bmatrix} 0 & 0 \\ -1 & 3 \\ 1 & -2 \end{bmatrix}$

If there are rows without leading ones, then the solution is impossible

If the matrix A is "tall" (i.e. $m > n$) then it is impossible to have a leading one in every row, so there is no right inverse.

<u>Matrix transpose:</u> $A = [a_{ij}]$,
$A^T = [a_{ji}]$
(rows of A become columns of A^T) and vice versa.

<u>Ex</u>:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

If $C = AB$, $c_{ij} = \sum_{k=1}^{n} a_{jk} b_{kj}$

$$c^T_{ij} = c_{ji} = \sum_{k=1}^{n} b_{kj} a_{jk}$$
$$= \sum_{k=1}^{n} b^T_{jk} a^T_{kj}$$

so $C^T = B^T A^T$

If $BA = I$ then $A^T B^T = I$
So if A is "long" ($m < n$) then A^T is "tall"

it has no right inverse, so
A has no left inverse.

<u>Def:</u> $\mathrm{rank}(A) = \#\{\text{leading ones in RREF}\}$.

$\mathrm{rank}(A) \leq \min\{m, n\}$ (because of the form of RREF)

<u>Theorem</u>: If $A \in M(m, n)$ then

(a) A has a right inverse
$\iff \mathrm{rank}(A) = m$

(b) A has a left inverse
$\iff \mathrm{rank}(A) = n$

[Thm 5.12] (c) A is invertible
$\iff$ A is square and $\mathrm{rank}(A) = m = n$.

<u>Proof</u>: (a) $[A \mid I] \longrightarrow$ Reduce
If RREF of A has zero rows
$\Rightarrow$ no solution.
If RREF has leading one in every row $\Rightarrow$ solutions possible.

(b) Follows from (a) by taking transpose.

(c) If A is square and $\mathrm{rank}(A) = m = n$
then $\mathrm{RREF}(A) = I$.

Further properties of Inverses:

1. [Thm 5.8] If A is invertible then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

Proof. $AA^{-1} = I_n$, $A^{-1}A = I_n$ — definition of inverse.

Same definition tells us that A is the inverse of A^{-1}, i.e. $(A^{-1})^{-1} = A$.

2. [Thm 5.9] If $B, A \in M(n,n)$; A, B invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: $BB^{-1} = I$, $B^{-1}B = I$, $AA^{-1} = I$, $A^{-1}A = I$

So $(AB)(B^{-1}A^{-1}) = A((BB^{-1})A^{-1}) = A(IA^{-1})$
$= AA^{-1} = I.$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}(AB)) =$
$= B^{-1}((A^{-1}A)B) = B^{-1}(IB) = B^{-1}B = I$ ✓

Homework: 5.2 : 1 bc, 2ad, 4bce, 8, 12