EXERCISES 5.2

#1 Determine the rank:

(b) $\begin{bmatrix} 4 & 6 \\ 6 & 9 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$ Reduce: $\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

one leading 1 $\Rightarrow$ rank = 1.

(C) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ Reduce: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

one leading 1 $\Rightarrow$ rank = 1

#2.

(a) $\begin{bmatrix} 7 & 9 & -8 & 4 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ Reduce: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4 leading ones $\Rightarrow$ rank = 4

(d) rank of an upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

is = the number of non-zero entries on the diagonal (a_{ii})

RREF will have 1's in places of nonzero entries, and 0's everywhere else.

#4 (b)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 1 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -14 & -4 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & -25 & -8 & -7 & 2 \\ 0 & 1 & -14 & -4 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.28 & -0.12 & 0.32 \\ 0 & 0 & 1 & 0.32 & 0.28 & -0.08 \\ 0 & 1 & 0 & 0.48 & -0.08 & -0.12 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.28 & -0.12 & 0.32 \\ 0 & 1 & 0 & 0.48 & -0.08 & -0.12 \\ 0 & 0 & 1 & 0.32 & 0.28 & -0.08 \end{array}\right]$$

4(c)

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & -6 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -1 & -4 & 1 & 1 & 0 \\ 0 & -3 & -12 & 2 & 0 & 1 \end{array}\right]$$

Since the rank of the original matrix is 2, there is no inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -3 & 1 \end{array}\right]$$

(e)

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 + R_1 \to R_1$
$2R_2 \to R_2$
$2R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 1 \\ -1 & 2 & -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 2 \end{array}\right]$$

$R_1 + R_2 \to R_2$
$R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 & 3 \end{array}\right]$$

$R_2 + R_1 \to R_1$
$\frac{1}{2}R_3 \to R_3$
$-R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{3}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{3}{2} \end{array}\right]$$

#8 (a)

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = [1]$$ — identity 1X1

(b) $$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$ — not an identity matrix

(c) $$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x & x \\ y & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ impossible since $x = 1$; $x = 0$
$y = 0$, $y = 1$
are incompartible.

#12.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

When $ad - bc \neq 0$, the matrix

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ is the inverse of the 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

when $ad - bc = 0$, if $a \neq 0$

then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is reduced to $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ no inverse.

if $a = 0$ then

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is reduced to $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ $(c \neq 0)$

or to $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ $(c = 0)$

In either case there is no inverse.

#8 $T: M(2,2) \to M(2,2)$:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Then
$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= \begin{pmatrix} a_1+a_2 & c_1+c_2 \\ b_1+b_2 & d_1+d_2 \end{pmatrix} = \begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}$$

$$= T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$$

(sums are transformed onto sums.)

and
$$T\left(r\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = T\left(\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}\right)$$

$$= \begin{pmatrix} ra & rc \\ rb & rd \end{pmatrix} = r\begin{bmatrix} a & c \\ b & d \end{bmatrix} = rT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

(scalar multiples are transformed onto scalar multiples.)

Therefore T is linear.

#15. (a) $N = \{ v \in V : T(v) = 0 \}$

If $v_1 \in N$, $v_2 \in N$ then $T(v_1) = 0$, $T(v_2) = 0$

Then $T(v_1 + v_2) = 0$ since T is linear $\Rightarrow v_1 + v_2 \in N$

If $r \in \mathbb{R}$, $v \in N$ then $T(rv) = rT(v) = 0$ since T is linear

$\Rightarrow rv \in N$

Finally $0 \in V \Rightarrow T(0) = 0$ since T is linear

so $0 \in N$

N is non-empty, closed for addition and mult. by scalar

$\Rightarrow$ N is a subspace.

(b) $R = \{ w \in W : w = T(v) \text{ for some } v \in V \}$

If $w_1, w_2 \in W$ then $w_1 = T(v_1)$, $w_2 = T(v_2)$

then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$ since T is linear

therefore $v_1 + v_2 \in R$.

If $r \in \mathbb{R}$, $w \in R$ then $rw = rT(v)$
$= T(rv)$
since T is linear

$\Rightarrow$ $rw \in R$

Finally $0 = T(0) \Rightarrow 0 \in R$

R is non-empty, closed for addition and multiplication by scalar

$\Rightarrow$ R is a subspace.

#17.

$$T : \mathbb{R}^3 \to \mathbb{P}_3$$

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = x^3+2x, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = x^2+x+1$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 2x^3-x^2$$

(a) $$T\left(\begin{bmatrix}5\\3\\-1\end{bmatrix}\right) = T\left(5\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}0\\1\\1\end{bmatrix} - 4\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= 5(x^3+2x) - 2(x^2+x+1) - 4(2x^3-x^2)$$

$$= -3x^3 + 2x^2 + 8x - 2$$

(b) $$T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = T\left(a\begin{bmatrix}1\\1\\1\end{bmatrix} + (b-a)\begin{bmatrix}0\\1\\1\end{bmatrix} + (c-b)\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= a(x^3+2x) + (b-a)(x^2+x+1)$$
$$+ (c-b)(2x^3-x^2)$$

$$= ax^3 + 2ax + (b-a)x^2 + (b-a)x + (b-a)$$
$$+ 2(c-b)x^3 - (c-b)x^2$$

$$= (a-2b+2c)x^3 + (-a+2b-c)x^2$$
$$+ (b+a)x + (b-a).$$

#18. $L : \mathbb{R}^2 \to \mathbb{R}^2$

$$L((1,1)) = (3,3)\ ; \quad L((1,-1)) = (3,-3)$$

Then

$$(a,0) = \frac{a}{2}(1,1) + \frac{a}{2}(1,-1)$$

$$(0,b) = \frac{b}{2}(1,1) - \frac{b}{2}(1,-1)$$

$$\Rightarrow (a,b) = \frac{a+b}{2}(1,1) + \frac{a-b}{2}(1,-1).$$

$$T((a,b)) = \frac{a+b}{2}T(1,1) + \frac{a-b}{2}T(1,-1)$$

$$= \frac{a+b}{2}(3,3) + \frac{a-b}{2}(3,-3)$$

$$= (3a, 3b) = 3(a,b).$$

#21 Suppose T <u>is</u> linear.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\1\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix} + \begin{bmatrix}5\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\4\end{bmatrix}$$

but $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix}$ – contradiction, so T cannot be linear.

MATH 262 EXERCISES 6.2

#1 (c) $s : \mathbb{R} \to \mathbb{R}$, $s(x) = \dfrac{e^x - e^{-x}}{2}$

$$s(x_1) = s(x_2) \Rightarrow \frac{e^{x_1} - e^{-x_1}}{2} = \frac{e^{x_2} - e^{-x_2}}{2}$$

let $e^{x_1} = u_1$, $e^{-x_1} = \frac{1}{u_1}$, $e^{x_2} = u_2$, $e^{-x_2} = u_2$

$$\Rightarrow u_1 - \frac{1}{u_1} = u_2 - \frac{1}{u_2}$$

$$u_1 - u_2 = \frac{1}{u_1} - \frac{1}{u_2} = \frac{u_2 - u_1}{u_1 u_2}$$

$\Rightarrow$ either $u_1 - u_2 = 0$, or $u_1 u_2 = -1$.

The latter is impossible since $u_1 u_2 > 0$

$\Rightarrow u_1 - u_2 = 0 \Rightarrow u_1 = u_2$

$\Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$ by taking ln.

$\Rightarrow s(x)$ is one-to-one.

Another way: $s'(x) = \dfrac{e^x + e^{-x}}{2} > 0$

$\Rightarrow s(x)$ is increasing

$\Rightarrow s(x_1) = s(x_2) \Rightarrow x_1 = x_2$

for otherwise $x_1 > x_2 \Rightarrow s(x_1) > s(x_2)$

$x_1 < x_2 \Rightarrow s(x_1) < s(x_2)$.

#1 (e) $L: \mathbb{R}^3 \to \mathbb{R}^3$; $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y-z \\ -x+2z \\ x+y+z \end{bmatrix}$ (2)

If $L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$ then

$$2x_1 + y_1 - z_1 = 2x_2 + y_2 - z_2$$
$$-x_1 + 2z_1 = -x_2 + 2z_2$$
$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2$$

$\Rightarrow$ $u = x_1 - x_2$, $v = y_1 - y_2$, $w = z_1 - z_2$ satisfy

$$\begin{cases} 2u + v - w = 0 \\ -u + 2w = 0 \\ u + v + w = 0 \end{cases}$$

Taking the sum of the first two equations we get the third equation, so the system of lin. equations will have a zero row in its reduced form
$\Rightarrow$ there are infinitely many solutions.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} t$$

is a solution.

For instance, $u = 2$, $v = -3$, $w = 1$,
So $x_2 = x_1 - 2$, $y_2 = y_1 + 3$, $z_2 = z_1 - 1$
gives the same values as x_1, y_1, z_1

The function L is then <u>not</u> one-to-one.

#2 (c)

$$\frac{e^x - e^{-x}}{2} = y$$

$$e^x = u; \quad u - \frac{1}{u} = 2y$$

$$u^2 - 2yu - 1 = 0$$
$$(u^2 - 2yu + y^2) - y^2 - 1 = 0$$
$$(u-y)^2 = 1 + y^2$$
$$u = y + \sqrt{1+y^2}$$
$$e^x = y + \sqrt{1+y^2}$$
$$x = \ln\left(y + \sqrt{1+y^2}\right)$$

There is a solution x for every $y \in \mathbb{R}$

$\Rightarrow$ the function $s(x)$ is onto $\mathbb{R}$.

(e)
$$\begin{cases} 2x + y - z = a \\ -x + 2z = b \\ x + y + z = c \end{cases} \Rightarrow a + b = c$$

Therefore the system of linear equations will not have solution for arbitrary a, b, c, for instance $a = 1$, $b = 1$, $c = 1$ is impossible.

Therefore
$$L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x + y - z \\ -x + 2z \\ x + y + z \end{pmatrix}$$

is <u>not</u> onto.

#8

(c) $$\frac{e^x - e^{-x}}{2} = y \Rightarrow x = \ln\left(y + \sqrt{1+y^2}\right)$$

so $$s^{-1}(x) = \ln\left(x + \sqrt{1+x^2}\right).$$

(e) L is <u>not</u> onto so there is no inverse function $\mathbb{R}^3 \to \mathbb{R}^2$.