EXERCISES 3.6

#1 $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(this is the first vector on the basis)

(b) $\begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}_B = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$.

$-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$

#2 $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(c) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(the zero vector is represented by the zero linear combination)

(d)
$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 5 \\ 0 & 1 & 1 & 6 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ -1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 2 & 3 & 8 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$r_1 = 2, \quad r_2 = 1, \quad r_3 = 5$

$2\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$

So $\begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

#4

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

(a) $$\begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix} = -3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \left[\begin{matrix} -3 & 0 \\ 0 & 4 \end{matrix}\right]_B = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 5 \\ 5 & 9 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 9\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \left[\begin{matrix} 1 & 5 \\ 5 & 9 \end{matrix}\right]_B = \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}.$$

#5 $$B = \{ x^2+1,\ x^2+x+2,\ 3x-1 \} = \{ p_1, p_2, p_3 \}$$

(b) $$\underbrace{x^2+x+2}_{p_2} - \frac{1}{3}\underbrace{(3x-1)}_{p_3} = x^2 + \frac{7}{3}$$

$$p_2 - \frac{1}{3}p_3 - \frac{7}{3}p_1 = \left(x^2+\frac{7}{3}\right) - \frac{7}{3}(x^2+1)$$

$$= \frac{4}{3}x^2$$

$$x^2 = \frac{3}{4}\left(p_2 - \frac{1}{3}p_3 - \frac{7}{3}p_1\right) = -\frac{7}{4}p_1 + \frac{3}{4}p_2 - \frac{1}{4}p_3$$

So $$[x^2]_B = \begin{bmatrix} -7/4 \\ 3/4 \\ -1/4 \end{bmatrix}$$

Note: Solving the system of linear equations

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$ provides another way to solve.

(c) $x^2+3x = \underbrace{(x^2+1)}_{P_1} + \underbrace{(3x-1)}_{P_3}$

$$[x^2+3x]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

#6. $B = \left\{ \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix} \right\}$

$$[A]_B = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$A = 3\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} + (-1)\begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$$

#2 (a)

$$AC = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 5 & -4 & 1 \end{pmatrix} \begin{bmatrix} 0 & 2 \\ -1 & -2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -3 & -4 \\ 7 & 18 \end{bmatrix}$$

(b) CA is undefined (inner dimensions do not match)

(c)

$$BC = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & -2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -1 & -6 \end{bmatrix}$$

(f)

$$B(A+I_3) = \begin{pmatrix} 1 & 3 & 4 \\ -2 & 1 & 0 \end{pmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 0 \\ 5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 26 & -3 & 7 \\ -5 & 2 & 2 \end{bmatrix}$$

#3.

$$\begin{cases} 2x_1 - x_2 - 3x_3 = 4 \\ x_1 + x_2 + x_3 = -2 \\ x_1 + 2x_2 + 3x_3 = 5 \end{cases}$$

$$A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} ; \quad b = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

then

$$\begin{bmatrix} 2 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -5 \end{bmatrix}$$

#6

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a^2 + bc = 1$

$ab + bd = 0 \Rightarrow b(a+d) = 0$

$ac + cd = 0 \Rightarrow c(a+d) = 0$

$bc + d^2 = 1$

If b or $c = 0$ (or both) then $bc = 0$

then $a^2 = 1$, $d^2 = 1$

$\Rightarrow A = \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}$, or $A = \begin{bmatrix} \pm 1 & 0 \\ c & \pm 1 \end{bmatrix}$

If $bc \neq 0$, then can find a^2, d^2

when $1 - bc \geq 0$ (i.e. $bc \leq 1$);

Since $a + d = 0$ the possibilities are

$a = \sqrt{1-bc}$, $d = -\sqrt{1-bc}$ or $a = -\sqrt{1-bc}$, $d = \sqrt{1-bc}$

$\Rightarrow A = \begin{bmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{bmatrix}$, $A = \begin{bmatrix} -\sqrt{1-bc} & b \\ c & \sqrt{1-bc} \end{bmatrix}$

for any b, c : $bc \leq 1$.