#2 $S = \{(2,4,2), (3,2,0), (1,-2,2)\}$.

Show that S is linearly independent.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 2 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{l} -2R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} -\frac{1}{4}R_2 \to R_2 \\ 3R_2+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we have leading ones in every column, the set S is linearly independent.

#4. $(2,2,6,0) = -(0,-1,0,1) + (1,2,3,3) + (1,-1,3,-2)$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 3 & -2 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$-R_1 \to R_1$

$-3R_2 + R_3 \to R_3$

$R_1 + R_4 \to R_4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF has ones in every column

$\Rightarrow$ linearly independent.

#5.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -3 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF has leading ones in every column

$\Rightarrow$ the set is linearly independent.

#7. $S = \{ x^2 + 2x + 3,\ x^2 + 2x + 1,\ x^2 + x + 2 \}$

Determine whether S is lin indep.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Partially reduced $\longrightarrow$ leading one on every column

$\Rightarrow$ lin. indep.

#8. Show that $\{\exp, \sin, \cos\}$ is lin. indep in $\mathcal{F}(\mathbb{R})$.

By contradiction: Suppose

$$c_1 \exp(x) + c_2 \sin(x) + c_3 \cos(x) = 0$$

not all $c_1, c_2, c_3 = 0$. (for all x.)

If $c_1 \neq 0$ then

$$\lim_{x \to +\infty} \{LHS\} = \pm\infty$$ (depending on the sign of c_1)

but $\lim_{x \to +\infty} \{RHS\} = 0$

— contradiction.

If $c_1 = 0$ then

$$c_2 \sin x + c_3 \cos x = 0 \quad \text{for all } x$$

$$\Rightarrow \quad c_2 \sin x = -c_3 \cos x$$

If $c_2 \neq 0$ then

$$\tan x = \frac{\sin x}{\cos x} = -\frac{c_3}{c_2}$$ — contradiction: $\tan x$ is not = constant

If $c_3 \neq 0$ then

$$\cot x = \frac{\cos x}{\sin x} = -\frac{c_2}{c_3}$$ — contradiction, $\cot x$ is not = constant