MATH 262

SOLUTIONS — PROBLEMS 2-3

#2 (c)
$$\begin{cases} x+2y+z=1 \\ x-y+2z=2 \\ x+8y-z=-1 \end{cases}$$

Matrix form (augmented matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 8 & -1 & -1 \end{array}\right]$$

$-R_1+R_2 \to R_2$
$-R_1+R_3 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & 1 & 1 \\ 0 & 6 & -2 & -2 \end{array}\right]$$

$-\frac{1}{3}R_2 \to R_2$
$-6R_2+R_3 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-2R_2+R_1 \to R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$
free

$$\begin{cases} x_1 = \frac{5}{3} - \frac{5}{3}t \\ x_2 = -\frac{1}{3} + \frac{1}{3}t \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/3 \\ 1/3 \\ 1 \end{bmatrix}$$

#3 (a) $\begin{cases} 2x - 4y + z = 1 \\ 4x + 2y - z = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 1 \\ 4 & 2 & -1 & 1 \end{array}\right]$$

For the sake of manual calculation, let's rearrange the order of columns (variables) in the order z, y, x:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 1 \\ -1 & 2 & 4 & 1 \end{array}\right]$$

$R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & -4 & 2 & 1 \\ 0 & -2 & 6 & 2 \end{array}\right]$

$-\frac{1}{2}R_2 \to R_2$

$4R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 10 & -3 \\ 0 & 1 & -3 & -1 \end{array}\right]$

free

$\begin{cases} x = t \\ y = -1 + 3t \\ z = -3 + 10t \end{cases} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$

(b) $\begin{cases} 3x + 6y + z = 0 \\ x + 2y + z = 2 \end{cases}$ Rearrange variables:

$$\begin{array}{ccc|c} (z) & (x) & (y) & \\ 1 & 3 & 6 & 0 \\ 1 & 1 & 2 & 2 \end{array}$$

$-R_1 + R_2 \to R_2$ $\quad \left[\begin{array}{ccc|c} 1 & 3 & 6 & 0 \\ 0 & -2 & -4 & 2 \end{array}\right]$

$-\frac{1}{2} R_2 \to R_2$

$-3R_2 + R_1 \to R_1$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$

free

$$\begin{cases} x = -1 - 2t \\ y = t \\ z = 3 \end{cases} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

#6 (a)

$$\begin{cases} x + 5y + 3z = -7 \\ -x - 5y - 2z = 4 \\ 2x + 10y + 5z = -11 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & -7 \\ -1 & -5 & -2 & 4 \\ 2 & 10 & 5 & -11 \end{bmatrix}$$

$R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 5 & 3 & -7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$R_2 + R_3 \to R_3$
$-3R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

$$\begin{cases} x = 2 - 5t \\ y = t \\ z = -3 \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

(b)

$$\begin{cases} w + y - 2z = 1 \\ 2w + x = 4 \\ w + x - y + z = 4 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 1 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 1 & -1 & 1 & 4 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$
$-R_1 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 4 & 2 \\ 0 & 1 & -2 & 3 & 3 \end{array}\right]$$

$-R_2 + R_3 \to R_3$
$-R_3 \to R_3$
$2R_3 + R_1 \to R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

free

$-4R_3 + R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{cases} w = -1 - t \\ x = 6 + 2t \\ y = t \\ z = -1 \end{cases} \Rightarrow \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

(c)

$$\begin{pmatrix} 2 & 1 & 2 & -3 & -2 & 5 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ -4 & -2 & 0 & 7 & 8 & -9 \end{pmatrix}$$

Note: To simplify the manual solution, we avoid normalization of the rows, leaving 2's in the leading positions.

$2R_2 + R_3 \to R_3$
$$\begin{bmatrix} 2 & 1 & 2 & -3 & -2 & 5 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 4 & 1 & 4 & 1 \end{bmatrix}$$

$-2R_2 + R_3 \to R_3$
$$\begin{bmatrix} 2 & 1 & 2 & -3 & -2 & 5 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_3 + R_2 \to R_2$
$-R_3 \to R_3$
$$\begin{bmatrix} 2 & 1 & 0 & -3 & -5 & 5 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$-R_2 + R_1 \to R_1$
$3R_3 + R_1 \to R_1$
$$\begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 8 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

free free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \\ -2 \end{bmatrix}$$

obtained by using -2 for x_5 and solving for the rest.

obtained by using 0's for the free variables.

obtained by using 2 as x_2 and solving for x_1

Note: To verify that the obtained solution works, plug back into the matrix form: the coefficient matrix

$$\begin{bmatrix} 2 & 1 & 2 & -3 & -2 \\ 0 & 0 & 2 & 1 & 1 \\ -4 & -2 & 0 & 7 & 8 \end{bmatrix}$$

times the columns $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \\ -2 \end{bmatrix}$

should give

$\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

respectively.

#8. (b) $\begin{bmatrix} 5 & -2 & 1 & 0 \\ 2 & 4 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 1 & 4 & 2 & 0 \end{bmatrix}$$

$-R_1+R_2 \to R_2$ $\begin{bmatrix} 1 & -2 & 5 & 0 \\ 0 & 6 & -3 & 0 \end{bmatrix}$

$\frac{1}{3}R_2 \to R_2$ $\begin{bmatrix} 1 & -2 & 5 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$

$R_2+R_1 \to R_1$ $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & -3 & 3 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$

$-R_1+R_2 \to R_2$
$-2R_1+R_3 \to R_3$
$-R_1+R_4 \to R_4$
$-R_1+R_5 \to R_5$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Note: choose a vector that gives 0 when multiplied by the second row, then choose the first entry so that it gives 0 when multiplied by the first row.

MATH 262

EXERCISES 3.1

#1 (a) $v_1 = (2, 1, -1)$, $v_2 = (3, 1, 1)$, $v_3 = (-4, 0, 6)$

$v = (1, 2, 6)$

$$(1,2,6) = (2,1,-1) + (3,1,1) + (-4,0,6) \quad \checkmark$$

(c') $v = (-12, -1, 9)$

$$r_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + r_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -12 \\ -1 \\ 9 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & -12 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 6 & 9 \end{array}\right)$$

$R_2 \leftrightarrow R_1$
$R_2 \leftrightarrow R_3$
$R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 1 & -4 & -10 \end{pmatrix}$$

$\frac{1}{2} R_2 \to R_2$
$-R_2 + R_3 \to R_3$

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -7 & -14 \end{pmatrix}$$

$-\frac{1}{7} R_3 \to R_3$
$-3R_3 + R_2 \to R_2$
$-R_2 + R_1 \to R_1$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{array}{l} r_1 = 1 \\ r_2 = -2 \\ r_3 = 2 \end{array}$$

$$(-12, -1, 9) = (2, 1, -1) - 2(3, 1, 1) + 2(-4, 0, 6).$$

#4.

$$p_1(x) = x+1, \quad p_2(x) = x^2+x, \quad p_3(x) = x^3+x^2$$

(c) $x^2+2x+1 = (x^2+x)+(x+1) = p_1(x)+p_2(x)$ ✓

(d) $(x+1)(x^2-x+1) = x^3+x^2-x^2-x+x+1 = x^3+1$

$p_3(x) - p_2(x)+p_1(x) = x^3+x^2-x^2-x+x+1$

$= x^3+1$ ✓

#6 Suppose $e^x = c_1 \sin x + c_2 \cos x$.

Choose $x = 2\pi n$, $n = 0, 1, 2 \ldots$

Then $c_1 \sin x + c_2 \cos x = c_1 \cdot 0 + c_2 \cdot 1$
$= c_2$

but $e^x = e^{2\pi n} \to \infty$ as $n \to \infty$.

Contradiction; representation

$e^x = c_1 \sin x + c_2 \cos x$ is impossible.

#9. (a) $\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$.

(b) $\cos(x) = c_1 + c_2 \sin^2 x$ is impossible,

because $\cos(x)$ has period 2π,
and $c_1+c_2 \sin^2 x$ has period π:

$x = 0 \Rightarrow 1 = c_1$
$x = \pi \Rightarrow -1 = c_1$ impossible.

#11.

$v = 5w_1 - 2w_2$, $w_1 = 8x_1 + x_2$, $w_2 = -3x_1 + 2x_2$

$$v = 5(8x_1 + x_2) - 2(-3x_1 + 2x_2)$$
$$= 40x_1 + 5x_2 + 6x_1 - 4x_2 \quad = 46x_1 + x_2$$