#2.
$$\begin{bmatrix} 5r+3s \\ -r+2s \\ s \end{bmatrix} = r\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

#3.
$$x = a+4b+c$$
$$y = 3a \quad -2c$$
$$z = \quad b+5c$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

#5.
(a) $a\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} a & 2a+b \\ 3a+2b+c & 4a+3b+2c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a=0$

$2a+b=0 \Rightarrow b=-2a \Rightarrow b=0$

$3a+2b+c=0 \Rightarrow c=-3a-2b=0$

$4a+3b+2c+d=0 \Rightarrow d=-4a-3b-2c=0$

(b) $\begin{bmatrix} a & 2a+b \\ 3a+2b+c & 4a+3b+2c+d \end{bmatrix} = \begin{bmatrix} 4, 9 \\ 16, 25 \end{bmatrix}$

$a=4$

$2a+b=9 \Rightarrow b=9-2a \Rightarrow b=1$

$3a+2b+c=16 \Rightarrow c=16-3a-2b \Rightarrow c=2$

$4a+3b+2c+d=25 \Rightarrow d=25-4a-3b-2c \Rightarrow d=2$

(c) No, because each step is obtained from the previous one in a manner of a logical implication. This way there is no guessing, or uncertainty.

#8. $S=\left\{ \begin{bmatrix} a, & b \\ c, & d \end{bmatrix} \in M(2,2) : 2a+b=0 \right\}$.

Show that S is closed under operations "+" and "·" in $M(2,2)$. Verify that S is a vector space with these ops.

(A1): $u=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$; $v=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$\left.\begin{array}{l} 2a_1+b_1=0 \\ 2a_2+b_2=0 \end{array}\right\} \Rightarrow 2(a_1+a_2)+(b_1+b_2)=0$$

Therefore $u+v \in S$

(A2): $r \in \mathbb{R}$, $u=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$2a_1+b_1=0 \Rightarrow 2(ra_1)+(rb_1)=0$

Therefore $ru \in S$

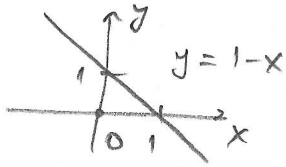
(A5): $u_0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $2\cdot 0+0=0 \Rightarrow u_0 \in S$ and $u+u_0=u$ for any $u \in S$

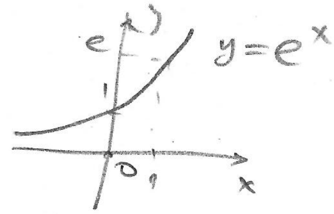
(A6): If $u=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $2a_1+b_1=0$ then $-u=\begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix}$ is s.t. $2(-a_1)+(-b_1)=0$ so $-u \in S$, and $u+(-u)=u_0$

All other axioms are properties that are satisfied for all matrices in $M(2,2)$, therefore for all matrices in S.

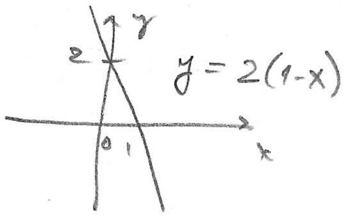
MATH 262 EXERCISES 1.7. (1)

#3 $f(x) = 1-x \;;\quad g(x) = e^x$

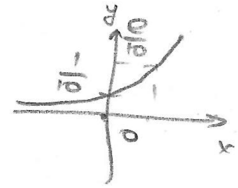




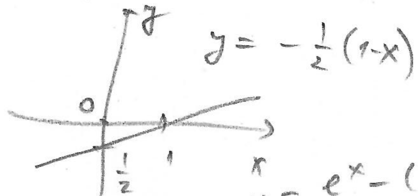
(a)



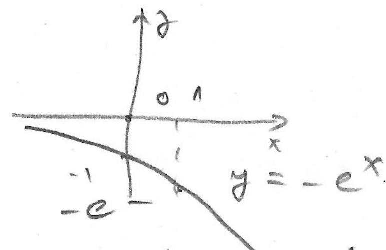
(c)



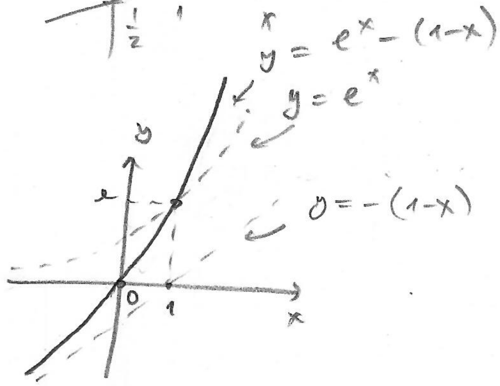
(b)



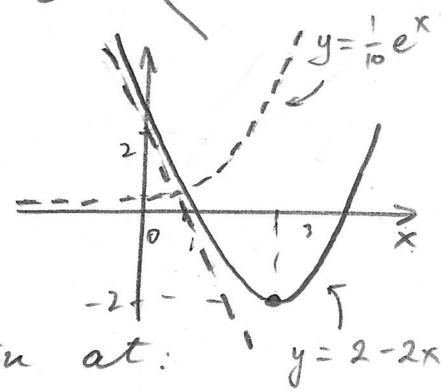
(d)



(e)



(f)



min at:

$$\frac{1}{10} e^x = 2$$

$$e^x = 20$$

$$x = \ln(20) \simeq 3.0$$

$$\left(2f + \frac{1}{10} g\right)(3) = 2 + 2(-2) = -2$$

#6.

$$\sin\left(x+\frac{\pi}{3}\right) = \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}$$

$$= \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

so $f(x) = \sin\left(x+\frac{\pi}{3}\right) = a\sin x + b\cos x$

holds when $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$.

#10 (a) $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + x_2, y_1 + y_2, \dots)$

$$r(x_1, x_2, \dots) = (rx_1, rx_2, \dots)$$

(A1) (A2) are satisfied by definition since the results of addition and mult are well-defined sequences.

(A3) (A4) are satisfied since add, mult are defined component-wise, so for instance

$$(y_1, y_2, \dots) + (x_1, x_2, \dots) = (y_1 + x_1, y_2 + x_2, \dots)$$
$$= (x_1 + y_1, x_2 + y_2, \dots) = (x_1, x_2, \dots) + (y_1, y_2, \dots)$$

(A5): the zero sequence $(0, 0, \dots)$ satisfies $(x_1, x_2, \dots) + (0, 0, \dots) = (x_1, x_2, \dots)$

(A6): the negative of a sequence $(x_1, x_2, \dots)$ is $(-x_1, -x_2, \dots)$ since $(x_1, x_2, \dots) + (-x_1, -x_2, \dots)$
$= (0, 0, \dots)$

(A7): $r((x_1, x_2, \dots) + (y_1, y_2, \dots))$

$$= r(x_1 + y_1, x_2 + y_2, \dots) = (rx_1 + ry_1, rx_2 + ry_2, \dots)$$
$$= r(x_1, x_2, \dots) + r(y_1, y_2, \dots).$$

(A8) $(r+s)(x_1, x_2, \dots) = ((r+s)x_1, (r+s)x_2, \dots)$

$$= (rx_1 + sx_1, rx_2 + sx_2, \dots)$$
$$= r(x_1, x_2, \dots) + s(x_1, x_2, \dots).$$

(A9), (A10) are verified in the same manner

The fact that there are inf. many components in each vector does not affect the verification of the axioms; the work is the same as for $\mathbb{R}^n$.

(b) Each sequence $(x_1, x_2, \dots)$ corresponds to the function $\overline{x}: \mathbb{N} \to \mathbb{R}$ defined by $\overline{x}(n) = x_n$.

Addition of functions $\sim$ Addition of seq.

Mult. by scalar of fns. $\sim$ Mult. by scalar of seq.

The space of sequences $(x_1, x_2, \ldots)$
is then represented by the space of functions
$\bar{x} : \mathbb{N} \to \mathbb{R}$.

(c), (d) $\bar{x} = (x_1, x_2, \ldots)$
$\bar{y} = (y_1, y_2, \ldots)$, $r \in \mathbb{R}$

$(\bar{x} + \bar{y})(n) = x_n + y_n \Rightarrow \bar{x} + \bar{y} = (x_1, x_2, \ldots) + (y_1, y_2 \ldots)$
$(r\bar{x})(n) = r x_n$ $\quad r\bar{x} = r(x_1, x_2, \ldots)$

#11. (a) $X = \{1, 2\}$, $Y = \{a, b, c\}$

3 choices for each of the elements in X
to form a function $f : X \to Y$
There are $3^2 = 9$ possible functions
from X to Y.

(b) $X = \{x_1, \ldots x_m\}$; $Y = \{y_1, \ldots y_n\}$

n choices of values to be assigned
for each of the element of X
therefore there are n^m choices
for functions $X \to Y$.

(c) If X or $Y = \emptyset$ then

$X \times Y = \emptyset$, so there is
only one relation between X and Y,
namely $\emptyset$, If $X = \emptyset$
then this relation is a function
which corresponds to counting $n^0 = 1$
even when $n = 0$.
If $X \neq \emptyset$ but $Y = \emptyset$
then the $\emptyset$ relation is not a function, $0^m = 0$.

#6.

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2,2) \ : \ a = b, \ b + 2c = 0 \right\}$$

Show S is a subspace of $\mathcal{M}(2,2)$.

$\begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix} \in S$ since $0 = 0$, $0 + 2\cdot 0 = 0$

so $S \neq \emptyset$.

If $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in S$, $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in S$ then $\left\{ \begin{array}{l} a_1 = b_1, \\ a_2 = b_2; \\ b_1 + 2c_1 = 0 \\ b_2 + 2c_2 = 0 \end{array} \right.$

then $\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in S$ since

$a_1 + a_2 = b_1 + b_2$ and

$b_1 + b_2 + 2(c_1 + c_2) = 0$.

Also if $r \in \mathbb{R}$ then

$\begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix} \in S$ since $\begin{array}{l} ra_1 = rb_1 \\ rb_1 + 2rc_1 = 0. \end{array}$

S is non-empty, closed under operations of addition and mult. by scalar $\Rightarrow$ S is a subspace in $\mathcal{M}(2,2)$.

#8

$$S = \{ f \in C[0,1] : \int_0^1 f(x)\,dx = 0 \}$$

Show S is a subspace of $C[0,1]$.

The zero function $f_0(x) = 0$, $x \in [0,1]$

satisfies $\int_0^1 f_0(x)\,dx = 0$

and is continuous at every pt. of $[0,1]$

Therefore S is non-empty.

If $f, g \in S$ then $\int_0^1 f(x)\,dx = 0$

$\int_0^1 g(x)\,dx = 0$

then $f+g$ is also continuous at every point of $[0,1]$,

and $\int_0^1 (f(x)+g(x))\,dx = 0$.

Thus, S is closed under addition of functions.

Also if $f \in S$ and $r \in \mathbb{R}$

then $rf \in C[0,1]$ (continuous at every point of $[0,1]$)

and $\int_0^1 r f(x)\,dx = 0$

$\Rightarrow rf \in S$, S is closed under multiplication by scalar.

#10. $U = \{(x,y) \in \mathbb{R}^2 : y = 3x\}$

$V = \{(x,y) \in \mathbb{R}^2 : y = 2x\}$

are both subspaces of $\mathbb{R}^2$

but $U \cup V = \{(0,0)\}$

so neither of U or V is contained in the other set.

#19. S, T - subspaces in V. Prove that $S \cap T$ is a subspace of V.

Proof: S, T - subspaces $\Rightarrow 0 \in S, 0 \in T$

$\Rightarrow 0 \in S \cap T \Rightarrow S \cap T \neq \emptyset$.

If $x \in S \cap T$, $y \in S \cap T$

then $x \in S, y \in S \Rightarrow x+y \in S$ (subspace)

also $x \in T, y \in T \Rightarrow x+y \in T$ (subspace)

$\Rightarrow \quad x+y \in S \cap T$

$S \cap T$ is closed under addition.

If $x \in S \cap T$, $r \in \mathbb{R}$ then

$x \in S,\ r \in \mathbb{R} \Rightarrow rx \in S$ (subspace)

$x \in T,\ r \in \mathbb{R} \Rightarrow rx \in T$ (subspace)

$\Rightarrow \quad rx \in S \cap T$

$\Rightarrow$ $S \cap T$ is closed under mult. by scalar.

#20

$$U = \{ (x,y) \in \mathbb{R}^2 : \ y = 3x \}$$

$$V = \{ (x,y) \in \mathbb{R}^2 : \ y = 2x \}$$

$$U \cup V = \{ (x,y) \in \mathbb{R}^2 : \ y = 3x \text{ or } y = 2x \}$$

is not a subspace:

$$(1,3),\ (1,2) \in U \cup V$$

but $(1,3) + (1,2) = (2,5) \notin U \cup V$

because in that case $\begin{array}{l} 5 \neq 2 \cdot 2 \\ 5 \neq 3 \cdot 2 . \end{array}$

Illustration.