#4. (a) Prove that $0+0=0$.

Proof: By Axiom (A5) (additive identity law)

for any $v \in V$ $\quad v+0=v$

Therefore $0+0=0$.

(b) If $v+v=v$ then $v=0$.

Proof: $v+v=v$

(A1) addition well-defined operation: + (A6) $\Rightarrow (v+v)+(-v) = v+(-v)$

(A4) assoc (A6) axiom of add. inverse $\Rightarrow v+(v+(-v)) = 0$

(A6) $\Rightarrow v+0=0$

(A5) $\Rightarrow v=0$.

#5. Prove that the zero element in V is unique.

Proof: By contradiction:

Suppose $v+0_1=v$ and $v+0_2=v$ where $0_1, 0_2$ are two different zero elements.

Using $v=0_2$ in the first equation,

$$0_2+0_1=0_2$$

Using $v=0_1$ in the second one,

$$0_1+0_2=0_1$$

But $0_2+0_1=0_1+0_2$ by (A3) $\Rightarrow 0_1=0_2$ contradiction.

#6: Prove that $r \cdot 0 = 0$.

Proof: Know from Thm 1.4 that $0 \cdot v = 0$ so the case $r = 0$ is contained in that statement.

Definition of zero: for any $v \in V$

$$v + 0 = v.$$

Let $v \in V$ and $r \neq 0$. Let $v = rw$ (so $w = \frac{1}{r} v$.)

Then $v + r \cdot 0 = r \cdot w + r \cdot 0$

$$= r(w + 0) = rw = v$$

(A7) (A.5)

Since 0 is unique in V, $r \cdot 0 = 0$.

#9. (a) Prove that $-0 = 0$.

(A6): $0 + (-0) = 0$

(A3): $(-0) + 0 = 0$

(A5) $(-0) = 0$, proved.

(b) If $-v = v$ then $v = 0$.

Proof: If $-v = v$ then

(A6) $v + v = 0$

(A10) $1 \cdot v + 1 \cdot v = 0$

(A8): $(1+1) \cdot v = 0$

$2 \cdot v = 0$

(A2): $\frac{1}{2} \cdot (2 \cdot v) = \frac{1}{2} \cdot 0$

(A9) + problem 6 $\left(\frac{1}{2} \cdot 2\right) v = 0 \Rightarrow 1 \cdot v = 0$

$\Rightarrow$ (A10) $v = 0$

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Prove:

#5 $(v-w)+w=v$

Proof: $(v-w)+w \overset{\text{def.}}{=} (v+(-w))+w$

$\underset{(A4)}{=} v+((-w)+w)$

$\underset{(A3)}{=} v+(w+(-w))$

$\underset{(A6)}{=} v+0$

$\underset{(A5)}{=} v$

Prove:

#6. $(v+w)-w=v$

Proof: $(v+w)-v \overset{\text{def.}}{=} (v+w)+(-w)$

$\underset{(A4)}{=} v+(w+(-w))$

$\underset{(A6)}{=} v+0 \underset{(A5)}{=} v$

Prove:

#7. $v-(w+x)=(v-w)-x$

Proof: $v-(w+x)= v+(-(w+x))$

Since $(w+x)+((-w)+(-x))$

$= (w+x)+((-x)+(-w))$

$= ((w+x)+(-x))+(-w)$

$= (w+(x+(-x)))+(-w)$

$= (w+0)+(-w) = w+(-w)=0$

we have $-(w+x)=(-w)+(-x)$

So $v-(w+x) = v+((-w)+(-x))$

$= (v+(-w))+(-x)$

$= (v-w)-x$

#8 Prove: $v-(w-x)=(v-w)+x$

$$v-(w-x)\overset{(def.)}{=} v+(-(w+(-x)))$$

$$= v+((-w)+(-(-x)))$$

$$\overset{(\#7)}{=} v+((-w)+x)$$

$$\underset{(Thm\ 1.5f)}{=} (v+(-w))+x$$

$$\underset{(A4)}{=} (v-w)+x.$$

(def)

#12. Prove: $(r-s)\,v=rv-sv.$

Proof: $(r-s)v=(r+(-s))v$

$$\underset{(A8)}{=} r\cdot v+(-s)v$$

$$= r\cdot v+((-1)\cdot s)v$$

$$\underset{(A9)}{=} r\cdot v+(-1)(sv)$$

$$\underset{(P.1.3.8)}{=} r\cdot v+(-sv)$$

$$\underset{(def)}{=} rv-sv$$

#13. Prove: $(v\neq 0,\ rv+w=sv+w)\Rightarrow r=s$

Proof: $rv+w=sv+w$

$$\underset{(Canc.\ Thm)}{\Rightarrow} rv=sv$$

$$\underset{(A1)}{\Rightarrow} rv+(-sv)=sv+(-sv)$$

$$\underset{(P1.4.12)}{\Rightarrow} rv+(-s)v=0$$

$$\Rightarrow (r-s)v=0$$

Problem 1.3.12 (Thm 1.5i)

By contradiction: If $r\neq s$ then $r-s\neq 0$

then $0=\frac{1}{r-s}\cdot 0=\frac{1}{r-s}\cdot((r-s)v)$

$$=\left(\left(\frac{1}{r-s}\right)\cdot(r-s)\right)\cdot v=1\cdot v=v$$

$\Rightarrow v=0$ contradiction.

Therefore we must have $r=s$

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EXERCISES 1.5

\#3 $\mathbb{R}^2 = \{ (v_1, v_2) : v_1 \in \mathbb{R}, v_2 \in \mathbb{R} \}$.

Define "+" and "·"
$$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1 - 1,\ v_2 + w_2)$$
$$r(v_1, v_2) = (rv_1 - r + 1,\ rv_2)$$

Prove that these operations satisfy the axioms of a vector space.

Proof: (A1) $(v_1 + w_1 - 1,\ v_2 + w_2) \in \mathbb{R}^2$; well-defined

(A2) $(rv_1 - r + 1,\ rv_2) \in \mathbb{R}^2$; well-defined.

(A3) $(w_1, w_2) + (v_1, v_2) = (w_1 + v_1 - 1,\ w_2 + v_2)$
$= (v_1 + w_1 - 1,\ v_2 + w_2) = (v_1, v_2) + (w_1, w_2)$

(A4) $((v_1, v_2) + (w_1, w_2)) + (x_1, x_2) = (v_1 + w_1 - 1,\ v_2 + w_2) + (x_1, x_2)$
$= ((v_1 + w_1 - 1) + x_1 - 1,\ v_2 + w_2 + x_2) = (v_1 + (w_1 + x_1 - 1) - 1,\ v_2 + (w_2 + x_2))$
$= (v_1, v_2) + ((w_1, w_2) + (x_1, x_2))$

(A5) The zero vector is $(1, 0)$:
$(v_1, v_2) + (1, 0) = (v_1 + 1 - 1,\ v_2 + 0) = (v_1, v_2)$.

(A6) $-(v_1, v_2) = (2 - v_1, v_2)$
then $(v_1, v_2) + (-(v_1, v_2)) = (v_1 + 2 - v_1 - 1,\ v_2 - v_2)$
$= (1, 0) = \vec{0}$

(A7) $r((v_1, v_2) + (w_1, w_2)) = r(v_1 + w_1 - 1,\ v_2 + w_2)$
$= (r(v_1 + w_1 - 1) - r + 1,\ rv_2 + rw_2)$
$= ((rv_1 - r + 1) + (rw_1 - r + 1) - 1,\ rv_2 + rw_2)$
$= r(v_1, v_2) + r(w_1, w_2)$

(A8) $(r+s)(v_1, v_2) = ((r+s)v_1 - (r+s) + 1,\ (r+s)v_2)$
$= (rv_1 + sv_1 - r - s + 1,\ rv_2 + sv_2)$
$= ((rv_1 - r + 1) + (sv_1 - s + 1) - 1,\ rv_2 + sv_2)$
$= (rv_1 - r + 1,\ rv_2) + (sv_1 - s + 1,\ sv_2)$
$= r(v_1, v_2) + s(v_1, v_2)$.

(A9) $r(sv) = r(sv_1 - s + 1, sv_2)$

$$= (r(sv_1 - s + 1) - r + 1, r(sv_2))$$
$$= ((rs)v_1 - rs + r - r + 1, (rs)v_2)$$
$$= ((rs)v_1 - rs + 1, (rs)v_2)$$

(A10) $1 \cdot v = (1 \cdot v_1 - 1 + 1, 1 \cdot v_2) = (v_1, v_2)$.

All axioms hold, $\mathbb{R}^2$ with the operations "+" and "·" is a vector space.

#5: $\mathbb{R}^2$ with operations defined by

$$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_1 + w_1 + v_2 + w_2)$$
$$r(v_1, v_2) = (rv_1, rv_1 + rv_2)$$

does not form a vector space.

Proof: By contradiction: If all axioms were satisfied, then

$$0 = 0 \cdot v = (0 \cdot v_1, 0v_1 + 0v_2) = (0, 0)$$

But $(v_1, v_2) + (0, 0) = (v_1, v_1 + v_2) \neq (v_1, v_2)$ unless $v_1 = 0$

a contradiction with Axiom 5. (additive identity.)

#10.

$P = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = v_2 + v_3\}$

Standard operations "+" and ".".

P is a vector space:

(A1), (A2) : if (v_1, v_2, v_3) and (w_1, w_2, w_3) are such that

$v_1 = v_2 + v_3$

$w_1 = w_2 + w_3$. then $v_1 + w_1 = (v_2 + w_2) + (v_3 + w_3)$

and $rv_1 = rv_2 + rv_3$

so $(v_1 + w_1, v_2 + w_2, v_3 + w_3) \in P$

$(rv_1, rv_2, rv_3) \in P$.

The zero vector $\vec{0} = (0,0,0)$

satisfies $0 = 0 + 0$, so $\vec{0} \in P$

Also if (v_1, v_2, v_3) is s.t. $v_1 = v_2 + v_3$

then $-v_1 = -v_2 - v_3$

$\Rightarrow (-v_1, -v_2, -v_3) \in P$.

All remaining properties of the vectors used in the axioms hold for all vectors in P because they hold for all vectors in the larger set $\mathbb{R}^3$.

Therefore P is a vector space.

#13.

Find a, b, c:

$a(2, 3, -1) + b(1, 0, 4) + c(-3, 1, 2) = (7, 2, 5)$

$3a + c = 2 \Rightarrow c = 2 - 3a$

$$\begin{cases} 2a + b - 3(2 - 3a) = 7 \\ -a + 4b + 2(2 - 3a) = 5 \end{cases}$$

$$\begin{cases} 11a + b = 13 \\ -7a + 4b = 1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$$

$$\Rightarrow c = -1$$

#14: Find a, b, c, d:

$$a(1,0,0,0,0) + b(1,1,0,0,0) + c(1,1,1,0,0) + d(1,1,1,1,0) = (8,5,-2,3,0)$$

Starting from the last component:

$0+0+0+0=0$ ✓

$0+0+0+d=3 \Rightarrow d=3$

$0+0+c+d=-2 \Rightarrow c=-5$

$0+b+c+d=5 \Rightarrow b=7$

$a+b+c+d=8 \Rightarrow a=3.$

#15: Show that it is impossible to find scalars a, b, c, d:

$$a(1,0,0,0,0) + b(1,1,0,0,0) + c(1,1,1,0,0) + d(1,1,1,1,0) = (8,5,-2,3,1)$$

The last component would produce the equation

$$0+0+0+0=1$$

which is impossible: $0 \neq 1$

Therefore, no matter what values are assigned to a, b, c, d, the vector equation cannot be satisfied.