EXERCISES 1.2

\# 7 Prove that the subsets are closed under addition of real numbers.

(a) $[5,\infty)$

Proof: Let $x \in [5,\infty)$, $y \in [5,\infty)$

$$\Rightarrow x \geq 5,\ y \geq 5 \Rightarrow x+y \geq 5+y,\quad 5+y \geq 5+5 = 10$$

$$\Rightarrow x+y \geq 10 \Rightarrow x+y \geq 5$$

$$\Rightarrow x+y \in [5,\infty).$$

(b) $\{3, 6, 9, 12 \ldots\}$

Proof: Any element of the set is $3k$, where k is an integer.

If $x = 3k_1$, $y = 3\cdot k_2$ then

$$x+y = 3k_1 + 3k_2 = 3(k_1+k_2)$$

$$\Rightarrow x+y \in \{3, 6, 9, 12 \ldots\}$$

(an integer multiple of 3.)

(c) $\{0\}$ Proof: $x=0, y=0 \Rightarrow x+y = 0+0 = 0.$

(d) $\emptyset$ Proof: $x \in \emptyset$ is false, so the implication

$$x \in \emptyset, y \in \emptyset \Rightarrow xy \in \emptyset$$

is true, by the truth table of "$\Rightarrow$".

#5 (a) A, B, C - sets

* $A \subseteq A$ true since $x \in A \Rightarrow x \in A$.
* $A \subseteq B$, $B \subseteq A \Rightarrow A = B$ true, this is the definition of set equality.

* $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$ true;

<u>Proof</u>: Suppose $x \in A$ then $x \in B$ since $A \subseteq B$.

But $B \subseteq C \Rightarrow (x \in B \Rightarrow x \in C)$

therefore $x \in C$

Since x is an arbitrary element of A, $A \subseteq C$.

In the case when $A = \emptyset$ when $x \in A$ is false, $\emptyset \subseteq C$ for any C.

* Either $A \subseteq B$ or $B \subseteq A$. False

Counter example: $A = \{1, 2\}$

$B = \{2, 3\}$

then neither $A \subseteq B$ $(1 \notin B)$

nor $B \subseteq A$ $(3 \notin A)$

(b) Interpret "+" as "$\cup$"

("to add two sets together means to take their union.")

$A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$ is true.

<u>Proof</u>: Suppose $A \subseteq B$ then for any $x \in A$

$x \in A \Rightarrow x \in B$.

If $x \in A \cup C$ then $x \in A$ or $x \in C$

then $x \in B$ or $x \in C$

then $x \in B \cup C$.

Therefore $A \cup C \subseteq B \cup C$.

(c) How about $x - (y - z) = (x - y) + z$?

$x - y \longleftrightarrow A \setminus B$ (set subtraction)

$x + y \longleftrightarrow A \cup B$ (set union)

$$A \setminus (B \setminus C) = \{ x : x \in A ,\ x \notin B \setminus C \}$$ negation

$$= \{ x : x \in A ,\ \overline{(x \in B ,\ x \notin C)} \}$$

$$= \{ x : x \in A ,\ (x \notin B \text{ or } x \in C) \}$$

$$= \{ x : (x \in A ,\ x \notin B) \text{ or } (x \in A ,\ x \in C) \}$$

$$= (A \setminus B) \cup (A \cap C)$$

So the analogy works if $C \subseteq A$,
but not otherwise.

Or how about using "$\cap$" instead of multiplication?

$x \le y ,\ z > 0 \Rightarrow xz \le yz$.

$A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$ — true.

<u>Proof</u>: Suppose $x \in A \Rightarrow x \in B$.

Let $x \in A \cap C$ then $x \in A$ and $x \in C$

then $x \in B$ and $x \in C$

$\Rightarrow x \in B \cap C$

Therefore $x \in A \cap C \Rightarrow x \in B \cap C$

$\Rightarrow A \cap C \subseteq B \cap C$.

So, there are no "negative sets" ~~that~~ would reverse the inclusion.

#3 (a) True or false:

$$S \cap T = T \cap S$$

$$S \cap T = \{x : x \in S, x \in T\} = \{x : x \in T, x \in S\}$$

(comma is used as logical "and") $= T \cap S$

so true.

(b) True or false:

$$(R \cap S) \cap T = R \cap (S \cap T)$$

$$R \cap S = \{x : x \in R, x \in S\}$$

$$(R \cap S) \cap T = \{x : x \in R \cap S, x \in T\}$$

$$= \{x : (x \in R, x \in S), x \in T\}$$

$$= \{x : x \in R, (x \in S, x \in T)\}$$

$$= \{x : x \in R, x \in S \cap T\}$$

$$= R \cap (S \cap T)$$

so true.

(c) True of false: $S \cap \emptyset = S$

Sometimes true: $\emptyset \cap \emptyset = \emptyset$

but generally false:

If $x \in S \cap \emptyset$ then $\underline{x \in S, x \in \emptyset}$

false since $x \in \emptyset$ is false

So $S \cap \emptyset$ does not contain any elements

$$S \cap \emptyset = \emptyset$$

And if $S \neq \emptyset$ then $S \cap \emptyset \neq \emptyset$.

#8 Which of the following are closed under multiplication of real numbers?

(a) $[5,\infty)$

YES: $x\in[5,\infty), y\in[5,\infty) \Rightarrow x\geq 5, y\geq 5$

$\Rightarrow xy \geq 25 \Rightarrow xy \geq 5$

$\Rightarrow xy \in [5,\infty)$

(b) $[0,1)$

YES: $x\in[0,1), y\in[0,1) \Rightarrow \begin{array}{l} 0\leq x<1 \\ 0\leq y<1 \end{array}$

$\Rightarrow xy\geq 0,\ xy<1$

$\Rightarrow xy\in[0,1)$.

(c) $(-1,0)$ Not closed: $-\frac{1}{2}\in(-1,0)$

but $\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)=\frac{1}{4}\notin(-1,0)$.

(d) $\{-1,0,1\}$

YES: $(-1)(-1)=1,\ (-1)0=0,\ (-1)1=-1$

$0(-1)=0,\ 0\cdot 0=0,\ 0\cdot 1=0$

$1\cdot(-1)=-1,\ 1\cdot 0=0,\ 1\cdot 1=1$

(e) $\{1,2,4,8,16\ldots\}=\{2^k;\ k=0,1,2..\}$

YES: $x=2^{k_1}, y=2^{k_2}$

$\Rightarrow xy=2^{k_1}2^{k_2}=2^{k_1+k_2}\in\{1,2,4..\}$

Since $k_1+k_2\in 0,1,2\ldots$

#9 (a) $S \ni \frac{1}{2}$, $S \ni \frac{1}{2}+\frac{1}{2}=1$

$S \ni 1+\frac{1}{2}=\frac{3}{2}$, $S \ni \frac{3}{2}+\frac{1}{2}=2$, etc

Any number in the form

$\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{n \text{ times}}$ must be in S

$S=\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots\right\}$

$=\left\{\frac{k}{2},\ k \text{ is a positive integer}\right\}$

is the smallest such set.

(b) $S \ni \frac{1}{2}$, $S \ni \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

$S \ni \frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$, $S \ni \frac{1}{8}\cdot\frac{1}{2}=\frac{1}{16}$, etc.

Any number in the form

$\frac{1}{2^k}$ must be in S

$S=\left\{\frac{1}{2^k} : k \text{ is a positive integer}\right\}$

$=\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\cdots\right\}$

is the smallest such set.