Thu., Apr. 24, 2008 **MATH 255A** Midterm Test 3

Name: (print) ______________________

Student ID No. : Solutions.

This test paper has 8 pages. The duration of the test is 1 hour 15 minutes. There are 6 questions in the main part (54 points) and one bonus question (6 points).

Your scores: (do not enter answers here)

1	2	3	4	5	6	bonus	total

Note: Write solutions in the space provided. If you run out of space you may use the back of the page. *Important: All answers should be justified. Show your work clearly and completely, explaining your steps.*

1. (6 points) Find and classify all relative extrema of the function $g(x) = 2x + \ln x$, if they exist.

Domain: $x > 0$ (for the ln to be defined).

$$g'(x) = 2 + \frac{1}{x}$$

$$2 + \frac{1}{x} = 0$$

$$\frac{1}{x} = -2$$

$$x = -\frac{1}{2}$$

not in the domain.

No critical points, no extrema.

(Another way: $y = 2x$ is an increasing function
$y = \ln x$ is an increasing function
$\Rightarrow$ $y = 2x + \ln x$ is increasing everywhere
$\Rightarrow$ no extrema.)

2. (8 points) For the function $f(t) = \sin^2 t$

(a) Find the third and the fourth order derivatives $f'''(t)$ and $f^{(4)}(t)$. Hint: use the identity $2\sin t\cos t = \sin(2t)$.

$$f'(t) = 2\sin t\cos t = \sin(2t)$$
$$f''(t) = 2\cos(2t)$$
$$f'''(t) = -4\sin(2t)$$
$$f^{(4)}(t) = -8\cos(2t)$$

(b) Find the values $f'''(t)$ and $f^{(4)}(t)$ for $t = \frac{\pi}{2}$ and $t = \pi$.

$$f'''\left(\frac{\pi}{2}\right) = -4\sin\pi = 0$$
$$f'''(\pi) = -4\sin 2\pi = 0$$
$$f^{(4)}\left(\frac{\pi}{2}\right) = -8\cos\pi = 8$$
$$f^{(4)}(\pi) = -8\cos 2\pi = -8.$$

3. (12 points) For the function $f(x) = x - x^3$

(a) Find the domain, the x- and y-intercepts.

Domain : all real x ($f(x)$ is a polynomial)

$x = 0 \Rightarrow f(x) = 0$
(y-intercept $y = 0$)

$y = 0 \Rightarrow x - x^3 = 0$

$x(1 - x^2) = 0$

$x = 0$ or $x^2 = 1$

$\underbrace{x = 0 \text{ or } x = 1 \text{ or } x = -1}_{x\text{-intercepts}}$.

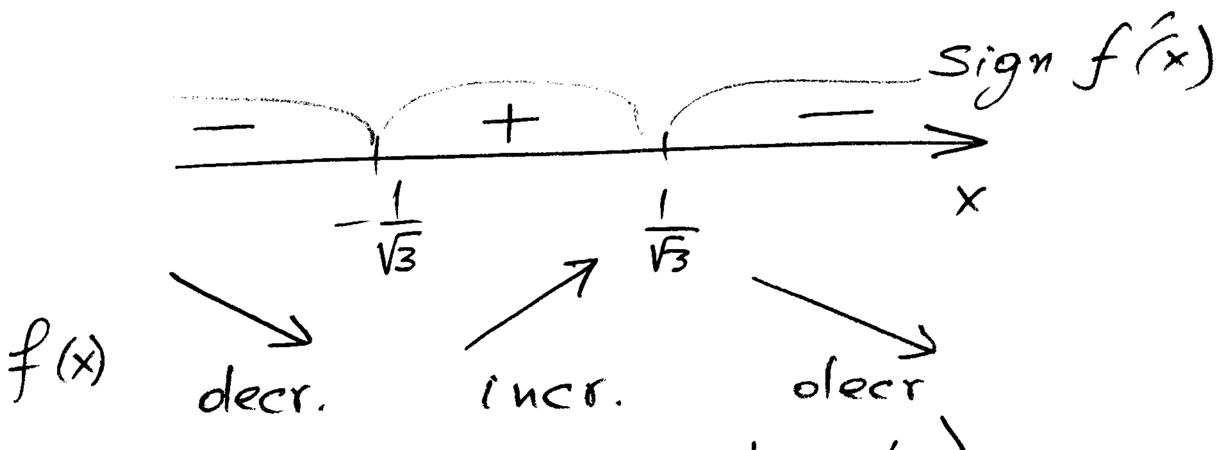
(b) Find the critical points and the intervals where $f(x)$ increases or decreases.

$f'(x) = 1 - 3x^2$

$1 - 3x^2 = 0$

$x^2 = \frac{1}{3}$

$x = \pm \frac{1}{\sqrt{3}}$ – critical points.



increases on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

decreases on $\left(-\infty, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \infty\right)$.

Continued...

(c) Find the inflection points and the intervals of convexity/concavity.

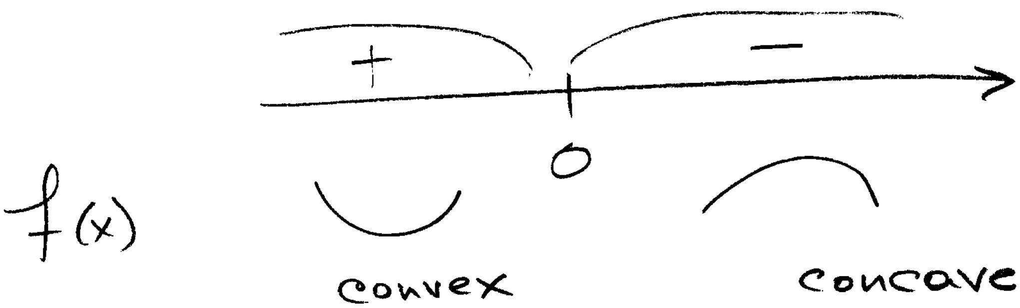
$f''(x) = 0$

$-6x = 0$

$\underline{\underline{x = 0}}$.

$f''(x) = -6x$;

sign of $f''(x)$



$x = 0$ inflection pt.

$f(-2) = 6$

$f(-1) = 0$

$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$

$f(0) = 0$

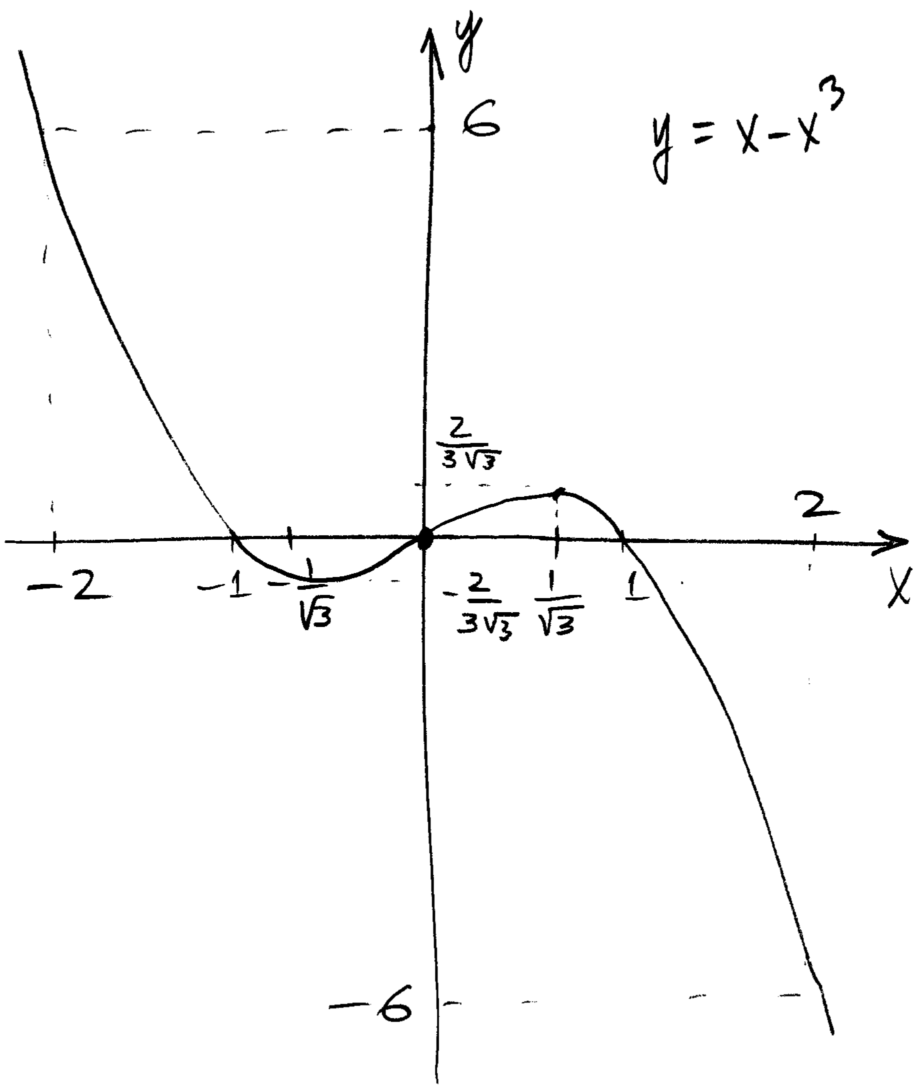
$f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$

$f(1) = 0$

$f(2) = -6$

$f(x)$ is odd.

(d) Using parts (a), (b) and (c) sketch the graph of $f(x)$.



4. (10 points) The number of people $P(t)$ (in hundreds) that are infected during week t of an epidemic is approximated by

$$P(t) = \frac{10 \ln t}{t} \qquad (t \geq 1).$$

(a) When will the number of people infected start to decline?

$$P'(t) = 10 \, \frac{\frac{1}{t} \cdot t - \ln t}{t^2}$$

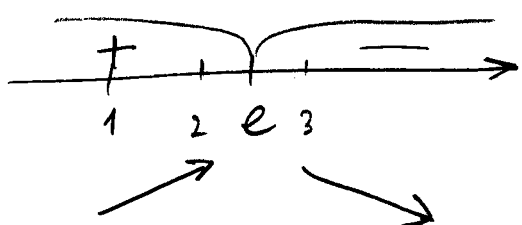
$$= 10 \, \frac{1 - \ln t}{t^2}$$

$$P'(t) = 0$$

$$1 - \ln t = 0$$

$$\ln t = 1$$

$$t = e \approx 2.71$$



sign $P'(t)$

$t = e$ point of max

$P(t)$ declines after $t = e$

(some time between the second and third week)

(b) Find the number of people infected when the epidemic is at its peak.

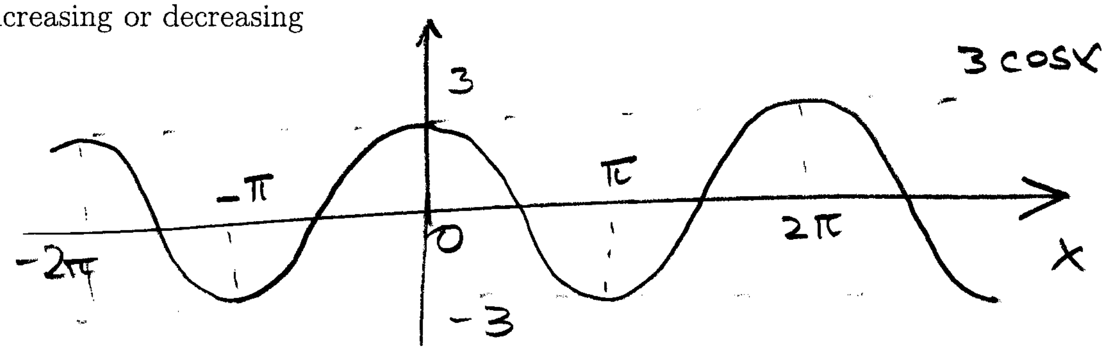
$$P(e) = \frac{10 \ln e}{e}$$

$$= \frac{10}{e} \approx 3{,}68 \quad \text{(hundreds)}$$

(368 people infected)

5. (8 points) Find the largest open intervals where the function $f(x) = 3\cos x$ is

(a) increasing or decreasing



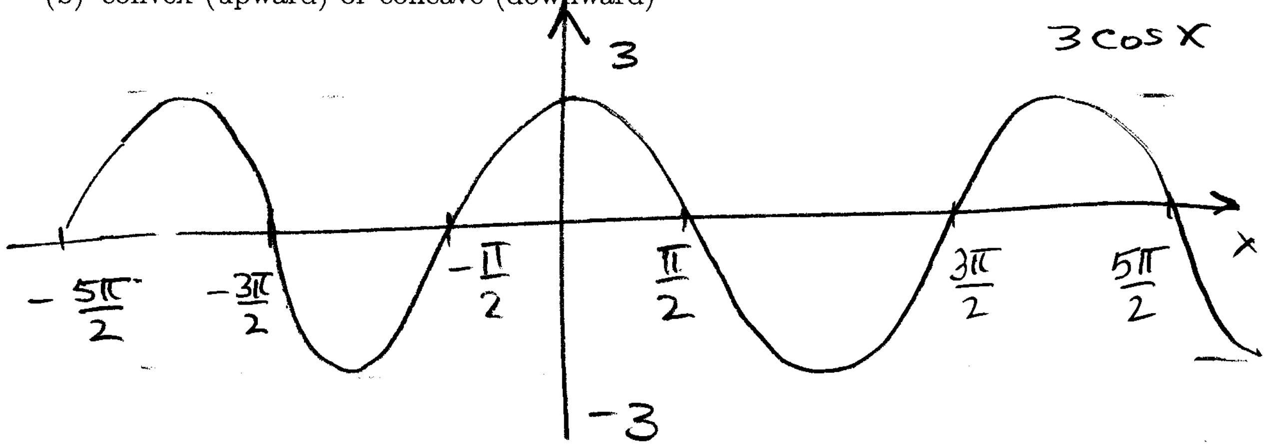
decreases on $\ldots (0, \pi), (2\pi, 3\pi) \ldots$

increases on $\ldots (-\pi, 0), (\pi, 2\pi), (3\pi, 4\pi) \ldots$

$(2\pi n, 2\pi(n+1))$, $n = 0, \pm 1, \pm 2 \ldots$

$(2\pi(n-1), 2\pi n)$, $n = 0, \pm 1, \pm 2 \ldots$

(b) convex (upward) or concave (downward)



Concave (downward) on $\ldots \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \ldots$

convex (upward) on $\ldots \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \ldots$

$\left(-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right)$, $n = 0, \pm 1, \pm 2 \ldots$

$\left(\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right)$, $n = 0, \pm 1, \pm 2 \ldots$

6. (10 points) Find the locations of all absolute extrema of the function $f(x)$ if they exist

$$f(x) = 2x + \frac{8}{x^2} + 1, \quad x > 0.$$

$$f'(x) = 2 - \frac{16}{x^3}$$

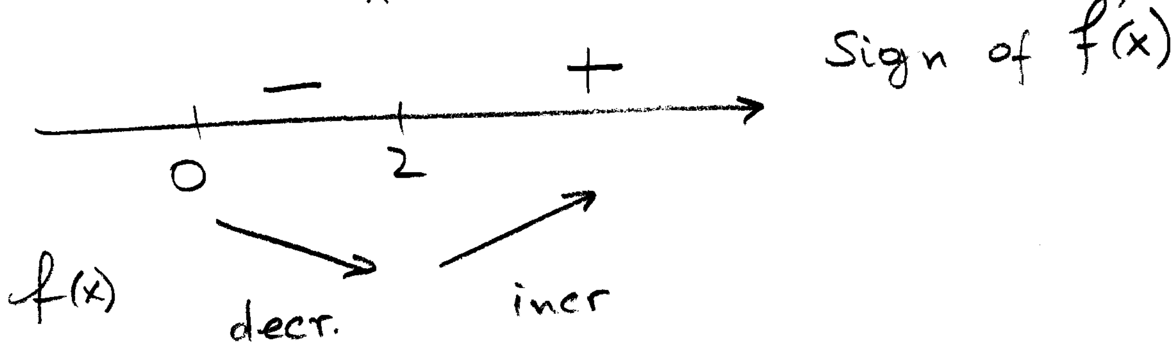
$$2 - \frac{16}{x^3} = 0$$

$$1 - \frac{8}{x^3} = 0$$

$$\frac{8}{x^3} = 1$$

$$x^3 = 8$$

$$x = 2$$



$x = 2$ – relative min

End points: $x = 0$: $\lim\limits_{x \to 0+} \left(\overset{\to 0}{2x} + \overset{\to \infty}{\frac{8}{x^2}} + 1\right) = +\infty$

$x = \infty$ $\lim\limits_{x \to \infty} \left(\underset{\to \infty}{2x} + \underset{\to 0}{\frac{8}{x^2}} + 1\right) = +\infty$

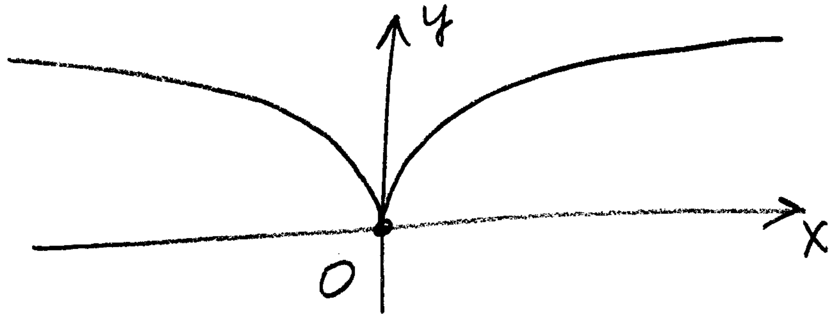
The fuction increases to ∞ at the end points and has the only rel. min. at $x = 2$

$\Rightarrow$ $x = 2$ – point of absolute min; no absolute max.

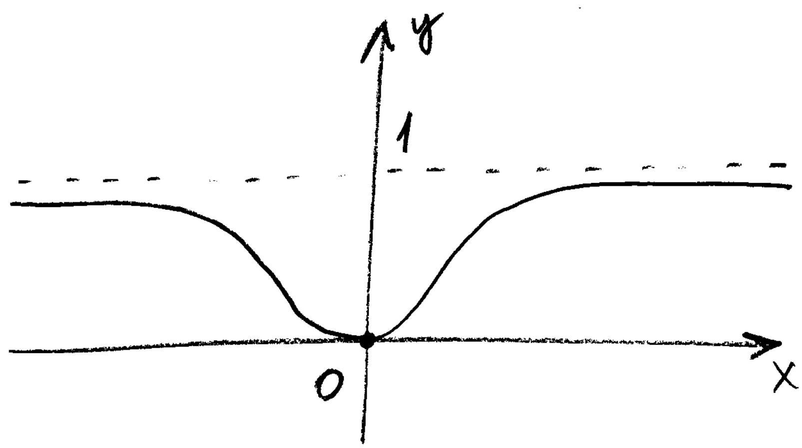
Continued...

7. (bonus problem: 6 points) A function $f(x)$ has a minimum at $x = 0$, is increasing for $x > 0$ and decreasing for $x < 0$. Is it true that $f(x)$ must be convex (upward)? If yes, justify your answer; if no, give an example of a function for which this is not true.

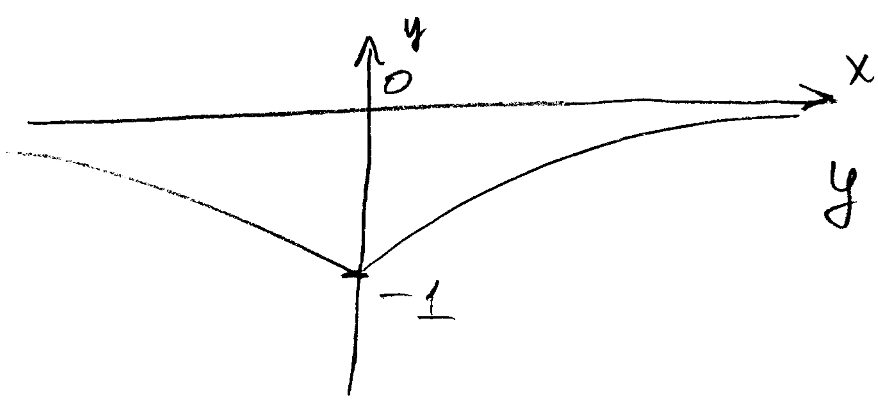
No. Here are a few examples:



$y = \sqrt{|x|}$



$y = \dfrac{x^2}{1+x^2}$



$y = -e^{-|x|}$.