Thu., Mar. 27, 2008 **MATH 255A** **Midterm Test 2**

Name: (print) ______________________

CSUN ID No. : *Solutions.*

This test paper has 8 pages. The duration of the test is 1 hour 15 minutes. There are 7 questions in the main part to the total of 52 points and one bonus question worth an extra 6 points.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Small print: Tests are closed books/notes. Graphing calculators may not be used. You should justify all answers properly. Partial credit will be given depending on how well you explain your solutions. Correct answers without explanations will receive zero points.

1. (5 points) The concentration of a drug in a patient's bloodstream t hours after it was injected is given by

$$A(t) = \frac{0.12\,t}{t^2+4} \qquad \text{(in [mg/l])}.$$

Find and interpret $\lim_{t\to\infty} A(t)$.

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \frac{t^2\left(\frac{0.12}{t}\right)}{t^2\left(1+\frac{4}{t^2}\right)} = \lim \frac{\frac{0.12}{t}}{1+\frac{4}{t^2}}$$

(factor out the highest power of t)

$\frac{0.12}{t} \to 0$, $\frac{4}{t^2} \to 0$

$$= \frac{0}{1} = 0.$$

As time goes on the amount of drug in the bloodstream decreases to zero.

2. (8 points) A particle is moving along a straight line; its position at time t is given by $s(t) = 10 - \frac{gt^2}{2}$ (g is a constant).

(a) Find the average speed of the particle over the interval $1 \leq t \leq 2$.

$$\text{Average speed} = \frac{\Delta s}{\Delta t}$$

$$\Delta s = s(2) - s(1) = \left(10 - \frac{4g}{2}\right) - \left(10 - \frac{g}{2}\right) = -\frac{3g}{2}$$

$$\Delta t = 1$$

$$\frac{\Delta s}{\Delta t} = -\frac{3g}{2}$$

(b) Use the definition of the derivative to find the instantaneous speed of the particle at $t = 1$.

Definition: $$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\cancel{10} - \frac{g(t+h)^2}{2}\right) - \left(\cancel{10} - \frac{gt^2}{2}\right)}{h}$$

$$= \lim_{h \to 0} \left(-\frac{g}{2}\right) \frac{(t+h)^2 - t^2}{h}$$

$$= \left(-\frac{g}{2}\right) \lim_{h \to 0} \frac{\cancel{t^2} + 2ht + h^2 - \cancel{t^2}}{h}$$

$$= \left(-\frac{g}{2}\right) \lim_{h \to 0} (2t + h) = -\frac{g}{2} \cdot (2t) = -gt$$

At $t = 1$: $s'(1) = -g$.

Continued...

3. (8 points) For the function

$$f(x) = \begin{cases} 0, & x < 0 \\ x(7-x), & 0 \le x \le 7 \\ 7x, & x > 7 \end{cases}$$

find all points where $f(x)$ is discontinuous. Find the left and the right limits of $f(x)$ at these points.

Transition points: $x=0$ and $x=7$.

$x=0$: $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 0 = 0$.

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \underbrace{x}_{\to 0}\underbrace{(7-x)}_{\to 7} = 0\cdot 7 = 0$

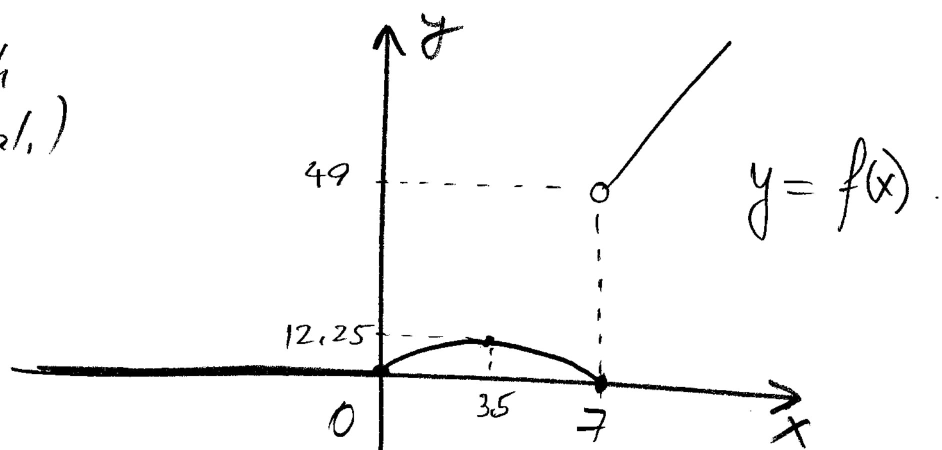
Right and left limits are the same
$\Rightarrow f(x)$ is continuous at $x=0$.

$x=7$: $\lim_{x\to 7-} f(x) = \lim_{x\to 7-} \underbrace{x}_{\to 7}\underbrace{(7-x)}_{\to 0} = 7\cdot 0 = 0$.

$\lim_{x\to 7+} f(x) = \lim_{x\to 7+} 7x = 7\cdot 7 = 49$.

Right and left limits are different
$\Rightarrow f(x)$ is discontinuous at $x=7$

Sketch
(optional.)



Continued...

4. (8 points) Find the derivatives of the functions

(a) $f(x) = \frac{5x+6}{\sqrt{x}}$ $= \frac{5x}{\sqrt{x}} + \frac{6}{\sqrt{x}} = 5\sqrt{x} + \frac{6}{\sqrt{x}}$

$= 5x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}$

$f'(x) = 5 \cdot \frac{1}{2} \cdot x^{\frac{1}{2}-1} + 6 \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{1}{2}-1}$

$= \frac{5}{2} x^{-\frac{1}{2}} - 3x^{-\frac{3}{2}}$

$= \frac{5}{2\sqrt{x}} - \frac{3}{x\sqrt{x}}$.

(b) $g(t) = (e^{-t} + 2t)^2$

$g'(t) = 2(e^{-t} + 2t)(-e^{-t} + 2)$

Chain rule: $f(u) = u^2, \quad f'(u) = 2u$

$u = h(t) = e^{-t} + 2t, \quad h'(t) = -e^{-t} + 2$

$g'(t) = f'(u)h'(t) = 2(e^{-t} + 2t)(-e^{-t} + 2)$.

5. (8 points) Find the limits, if they exist. If they don't explain why.

(a) $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$

$$\left(\text{plug in} \Rightarrow \frac{\sqrt{16} - 4}{16 - 16} = \frac{0}{0} = ? \right)$$

$$\frac{\sqrt{x} - 4}{x - 16} = \frac{(\sqrt{x} - 4)(\sqrt{x} + 4)}{(x - 16)(\sqrt{x} + 4)} = \frac{x - 16}{(x - 16)(\sqrt{x} + 4)}$$

$$= \frac{1}{(\sqrt{x} + 4)} \quad (x \neq 16)$$

$$\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} = \lim_{x \to 16} \frac{1}{\sqrt{x} + 4} = \frac{1}{\sqrt{16} + 4} = \frac{1}{8}.$$

(b) $\lim_{x \to 0} \frac{|x|}{x}$

$$\left(\text{plug in} \Rightarrow \frac{|0|}{0} = \frac{0}{0} = ? \right)$$

$$\frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x > 0 \\ -\frac{x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

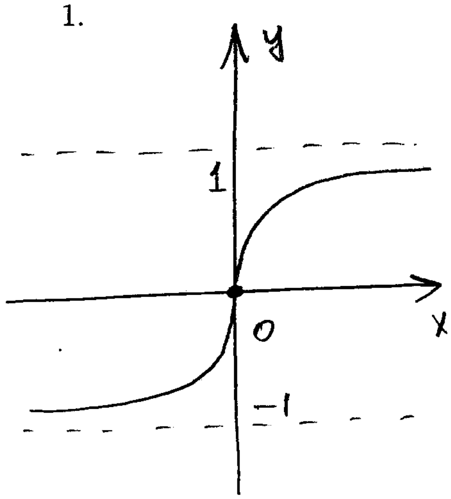
$$\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} -1 = -1$$

$$\lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} 1 = 1$$

Right and left limits are different $\Rightarrow$ the limit $x \to 0$ Does Not Exist.

6. (6 points) In this problem you are given the graphs of six functions. Under each graph indicate for which values of x the function is (a) discontinuous (b) not differentiable. (You do not need to explain why.) (derivative does not exist.)

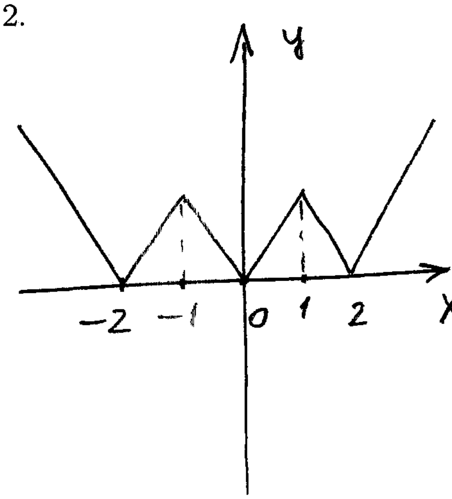
1.



(a) none

(b) $x = 0$

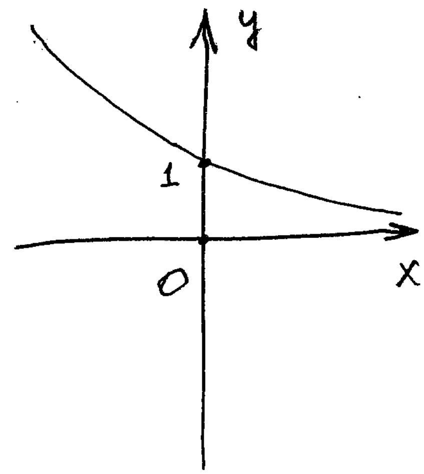
2.



(a) none

(b) $x = -2, -1, 0, 1, 2$

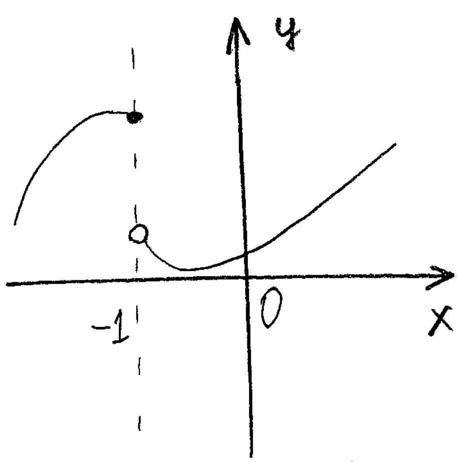
3.



(a) none

(b) none

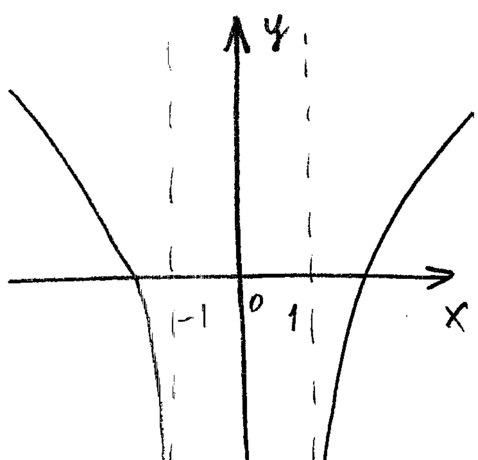
4.



(a) $x = -1$

(b) $x = -1$

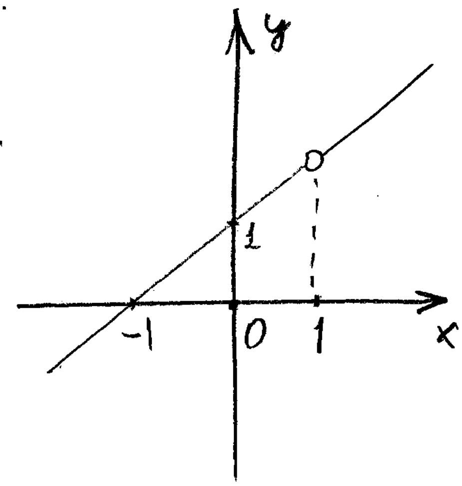
5.



(a) $-1 \leq x \leq 1$

(b) $1 \leq x \leq 1$

6.



(a) $x = 1$

(b) $x = 1$

7. (9 points) Let $f(x) = \sqrt{2x-1}$. $= g(h(x))$

(a) Find the derivative $f'(x)$.

$g(u) = \sqrt{u}$; $g'(u) = \frac{1}{2\sqrt{u}}$

$u = h(x) = 2x-1$; $h'(x) = 2$

$$f'(x) = g'(u)h'(x) = \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2} =$$

$$= \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2x-1}}$$

(b) Find the equation of the tangent line to the graph $y = f(x)$ at $x = 1$.

$$y - y_0 = m(x - x_0)$$

$x_0 = 1$, $y_0 = f(x_0) = \sqrt{2 \cdot 1 - 1} = 1$

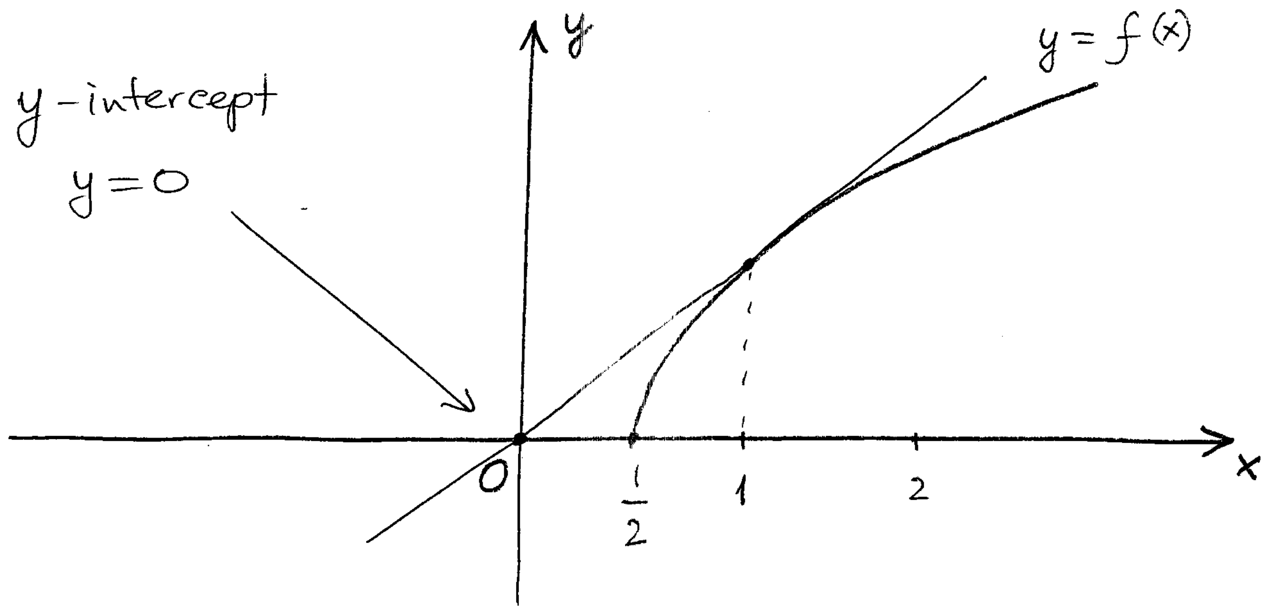
$$m = f'(x_0) = \frac{1}{\sqrt{2 \cdot 1 - 1}} = 1$$

$$y - 1 = 1 \cdot (x - 1)$$

$$y - 1 = x - 1$$

$$y = x$$

(c) Sketch the tangent line from part (b) on the graph below. Find the y-intercept.



8. (bonus: 6 points) Christie runs out of flour for the cake she is baking so she drives to the nearest grocery store at the speed of 60 mph. When she arrives there she finds that the store is closed, so she drives back at the speed of 40 mph. What was Christie's average speed for the trip?

$$\text{Average speed} = \frac{\Delta s}{\Delta t} = \frac{\text{Distance traveled}}{\text{time}}$$

Say the store is ℓ miles from Christie's home. Then $\Delta s = 2\ell$

$$\text{time} = \underbrace{\frac{\ell}{60}}_{\text{to the store}} + \underbrace{\frac{\ell}{40}}_{\text{from the store.}}$$

$$\frac{\Delta s}{\Delta t} = \frac{2\ell}{\frac{\ell}{60} + \frac{\ell}{40}} = \frac{2}{\frac{1}{60} + \frac{1}{40}} = \frac{20}{\frac{1}{6} + \frac{1}{4}} = \frac{20 \cdot 24}{10}$$

$$= 2 \cdot 24 = 48.$$

[mph.]

The end.