Thu, Nov 21, 2019 **MATH 255A Midterm 3, Version 1** Prof. V. Panferov

Name: (print) __________________________

CSUN ID No. : Solutions.

This test includes 7 questions, on 8 pages. Last page is a formula sheet. The perfect score is 40 points. The last problem included a 4 points bonus. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books and notes. No electronic devices are allowed, except an approved model of graphing calculator. All cell phones must be off and put away completely for the duration of the exam. Show all your work.

1. (6 points) Given

$$f(x) = \frac{2x}{x+1}, \qquad x_0 = 0.$$

(a) Find the derivative $f'(x)$.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$f'(x) = \frac{2\cdot(x+1) - 2x\cdot 1}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(b) Find the value $f'(x_0)$.

$$f'(0) = \frac{2}{(0+1)^2} = 2$$

(c) Find the tangent line to the graph $y = f(x)$ at $x = x_0$. Sketch a graph to illustrate the relationship between the graph and the tangent line.

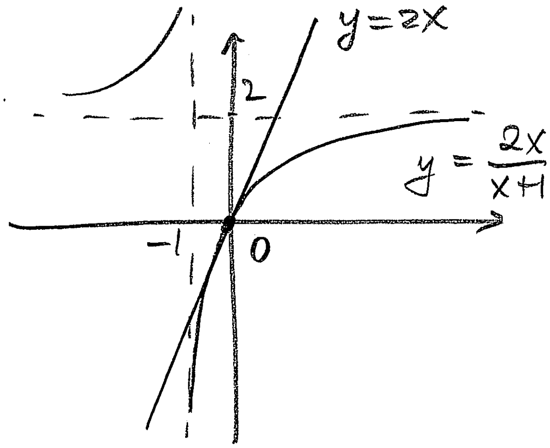
$$\left.\begin{array}{l} x_0 = 0 \\ y_0 = f(x_0) = 0 \\ m = f'(x_0) = 2 \end{array}\right\} \Rightarrow$$

tangent line eq.

$$y - y_0 = m(x - x_0)$$

$$y = 2(x - 0) = 2x$$

$$y = 2x$$



2. (6 points) Given the function $y = 100\,x^{-2.5}$.

(a) Find the sensitivity and elasticity of y to x when $x = 10$.

$$f(x) = 100\,x^{-2.5}\;;\quad a = 10$$

$$f'(x) = 100 \cdot (-2.5) \cdot x^{-2.5-1} = -250\,x^{-3.5}$$

$$S = f'(a) = -250 \cdot 10^{-3.5} = -0.07906$$

$$E = \left.\frac{f'(x) \cdot x}{f(x)}\right|_{x=a} = \frac{\cancel{100}\,(-2.5)\,\cancel{x^{-3.5}} \cdot \cancel{x}}{\cancel{100} \cdot \cancel{x^{-2.5}}} = -2.5$$

(b) If the value x is known as approximately 10 with percent error $\pm 2\%$, what is the corresponding percent error in the computed value y?

$$\delta y \cong E\,\delta x$$

$$\delta y \cong -2.5 \cdot (\pm 2\%) = \mp 5\%$$

The estimate of the error is $\pm 5\%$.

3. (4 points) Use implicit differentiation to find $\frac{dy}{dx}$ at the point $(x_0, y_0) = (1, 1)$:

$$y^2 + y - x^5 = 1.$$

$$\frac{d}{dx}\left(y^2 + y - x^5\right) = \frac{d}{dx}(1)$$

Here $y = y(x)$, and $y^2 = (y(x))^2$ - composite function.

$$\frac{d}{dx} y^2 = \underbrace{2y}_{\text{outer}} \cdot \underbrace{y'(x)}_{\text{inner}}$$

$$\frac{d}{dx} y = y'(x)$$

$$\frac{d}{dx} x^5 = 5x^4 \; ; \qquad \frac{d}{dx}(1) = 0$$

$$\Rightarrow \quad 2yy' + y' - 5x^4 = 0$$

$$y'(2y+1) = 5x^4$$

$$y' = \frac{5x^4}{2y+1}$$

$$y'\Big|_{\substack{x=1\\y=1}} = \frac{5}{2+1} = \frac{5}{3}.$$

4. (6 points) The height of the tide at a beach is given by the function

$$H(t) = 0.8 \sin\left(\frac{\pi}{6}(t-1)\right) + 2.2 \quad \text{[ft]}$$

where t is measured in hours after midnight.

(a) Find the height of the tide at 10am.

$$H(10) = 0.8 \sin\left(\frac{\pi}{6}(10-1)\right) + 2.2$$
$$= 0.8 \sin\left(\frac{9\pi}{6}\right) + 2.2$$
$$= 0.8 \sin\left(\frac{3\pi}{2}\right) + 2.2$$
$$= 0.8(-1) + 2.2 = 1.4 \text{ [ft]}.$$

(b) Compute the derivative $\frac{dH}{dt}$.

$$\frac{dH}{dt} = \left(0.8 \sin\left(\frac{\pi}{6}(t-1)\right) + 2.2\right)'$$
$$= 0.8 \cos\left(\frac{\pi}{6}(t-1)\right) \cdot \frac{\pi}{6} + 0$$
$$= \frac{0.8\pi}{6} \cos\left(\frac{\pi}{6}(t-1)\right).$$

(c) Find the value $\left.\frac{dH}{dt}\right|_{t=10}$ and interpret. Is the height of tide on the increase or on the decrease at 10am?

$$\left.\frac{dH}{dt}\right|_{t=10} = \frac{0.8\pi}{6} \cos\left(\frac{\pi}{6} \cdot 9\right) = \frac{0.8\pi}{6} \cos\left(\frac{3\pi}{2}\right)$$
$$= 0 \text{ [ft/hour]}$$

10 am corresponds to a critical point of $H(t)$, when the instantaneous rate of change $= 0$.
So, locally at 10 am the function $H(t)$ is neither incr. or decr, but has a local minimum (since 1.4 is the least possible value of $H(t)$.)

5. (6 points) Given

$$f(x) = e^{-x} - e^{-2x}$$

(a) Compute the derivative $f'(x)$.

$$f'(x) = -e^{-x} - (-2)e^{-2x}$$
$$= -e^{-x} + 2e^{-2x}$$

(b) Find all critical points of $f(x)$.

$$f'(x) = 0 \iff -e^{-x} + 2e^{-2x} = 0$$
$$e^{-x} = 2e^{-2x}$$

[mult. both sides by e^{2x} to simplify]

$$e^{-x} \cdot e^{2x} = 2e^{-2x} \cdot e^{2x}$$
$$e^{2x-x} = 2\underbrace{e^{-2x+2x}}_{=e^0=1}$$
$$e^x = 2$$
$$\ln e^x = \ln 2$$
$$x = \ln 2 \approx 0.69$$ — critical point

(c) Find the global maximum and the global minimum (if they exist) for $f(x)$ on the interval $[0, \infty)$.

x	0	$\ln 2$	∞
y	0	0.25	0

$\Rightarrow$ [global max 0.25 when $x = \ln 2$; global min 0 when $x = 0$.]

$$f(\ln 2) = e^{-\ln 2} - e^{-2\ln 2}$$
$$= (e^{\ln 2})^{-1} - (e^{\ln 2})^{-2} = 2^{-1} - 2^{-2}$$
$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = 0.25$$

(calculator OK too!)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{-x} - \lim_{x\to\infty} e^{-2x} = 0 - 0$$
$$= 0$$

Continued...

6. (6 points) Sketch a graph of the function

$$f(x) = x^3 - 12x + 16$$

showing all critical points and inflection points. All work must be shown.

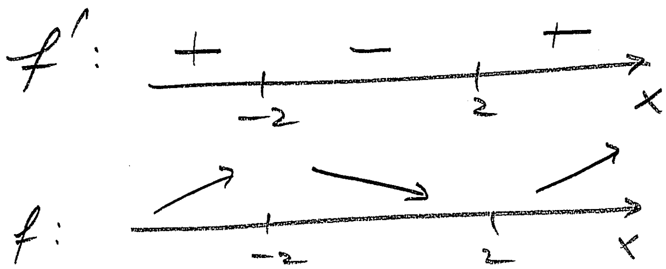
$f'(x) = 3x^2 - 12$

$f'(x) = 0 \iff 3x^2 - 12 = 0$

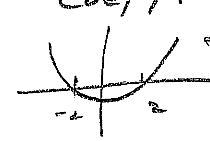
$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$ — critical pts.

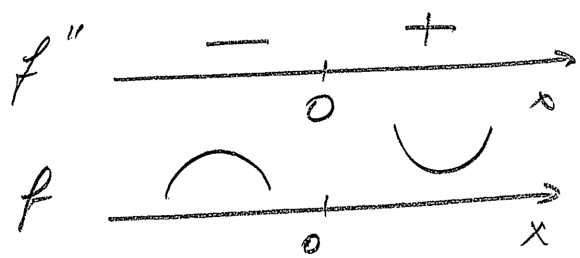


since $f'(x)$ is a parabola with positive coefficient at x^2-term.

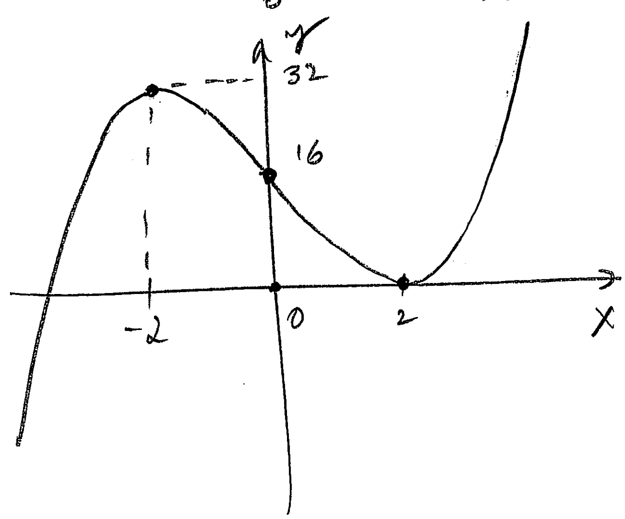


$f''(x) = 6x$

$f''(x) = 0 \iff x = 0$



$\Rightarrow$ $x = 0$ is an inflection pt.



$f(0) = 16$

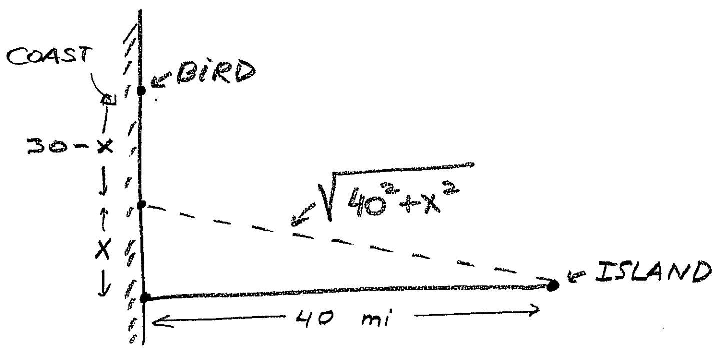
$f(-2) = 32$

$f(2) = 0$

Continued...

7. (6+4 points) A bird is trying to reach an island located 30 miles south and 40 miles east from the coast (see figure). Suppose that it takes for the bird 30% more energy to fly one mile over the water than to fly one mile over the land. The bird flies $30 - x$ miles along the coast and then straight to the island over the water. Find the value of x that minimizes the total amount of energy used.

(6 points to set up the function to minimize, 4 points bonus for finding the value x.)



The energy to fly over land: $(30-x)E_1$

E_1 - amount of energy to fly one mile over land

The energy to fly over water: $\sqrt{40^2+x^2} \cdot 1.3E_1$

Since to fly one mile over water takes $E_1 + 30\% E_1 = E_1 + 0.3E_1 = 1.3E_1$ units of energy.

Total energy: $E(x) = (30-x)E_1 + 1.3E_1\sqrt{40^2+x^2}$

Since the value of E_1 does not affect location of min, can minimize

$$f(x) = (30-x) + 1.3\sqrt{40^2+x^2}.$$

Solve: $f'(x) = -1 + 1.3\dfrac{x}{\sqrt{40^2+x^2}}$

$1 = 1.3\dfrac{x}{\sqrt{40^2+x^2}}$

$\sqrt{40^2+x^2} = 1.3x$

$40^2+x^2 = 1.69x^2$

$40^2 = 0.69x^2$

$x^2 = \dfrac{40^2}{0.69}$

$x = \pm\sqrt{\dfrac{40^2}{0.69}} \approx \pm 48.15$

Since the values ± 48.15 are outside the interval $[0, 30]$, the minimum must be attained at one of the end points.

Since $f'(0) = -1 < 0$, the minimum must occur at $x = 30$

The end.

$f(30) = 65$

$E(30) = 65 \cdot E_1$ — least amount of energy.

Table of formulas

Derivative Rules:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$u(x) \pm v(x)$	$u'(x) \pm v'(x)$
$cu(x)$	$cu'(x)$
e^{mx}	me^{mx}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
uv	$u'v + uv'$
$\frac{u}{v}$	$\frac{u'v-uv'}{v^2}$
$g(u(x))$	$g'(u(x))u'(x)$

Trigonometry:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Linear approximation: For x near a:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

Quadratic (second-order) approximation: For x near a:

$$f(x) \approx Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Sensitivity and Elasticity: If $y = f(x)$, $x = a$ – exact value,

$$\Delta y \approx S\,\Delta x; \qquad S = f'(a), \qquad \Delta y,\ \Delta x \text{ – absolute errors.}$$

$$\delta y \approx E\,\delta x; \quad E = \frac{f'(a)\,a}{f(a)}, \quad \delta y = \frac{\Delta y}{f(a)} \cdot 100\%, \quad \delta x = \frac{\Delta x}{a} \cdot 100\% \text{ – percentage errors.}$$

The end.