Thu, Oct 17, 2019 **MATH 255A Midterm 2, Version 1** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test includes 7 questions (40 points in total), on 7 pages. Page 8 is a formula sheet. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. No electronic devices except an approved model of graphing calculator. All cellphones must be off and put away completely for the duration of the exam. Show all your work.

1. (4 points) Estimate the derivative $f'(a)$ numerically, using a table of values:

$$f(x) = 4^x, \quad a = 0.$$

Compare the obtained estimate with the exact value obtained using Derivative Rules.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{4^h - 1}{h}$$

$h = L_1$	$(4^{L_1}-1)/L_1$
0.1	1.487
0.01	1.3959
0.001	1.3873
0.0001	1.3864
−0.0001	1.3862
−0.001	1.3853
−0.01	1.3767
−0.1	1.2945

Estimate:

(half-way between 1.3864 and 1.3862;)

limit $\simeq$ 1.3863

Exact: $f'(x) = (\ln 4) \cdot 4^x$

$f'(0) = \ln 4 \simeq 1.38629$

(accurate to 4 decimal places.)

2. (6 points) Find the limits *exactly*. Show work. If the limits do not exist provide reasons for your answer.

(a) $\lim_{x\to 0} \frac{|2x|}{x}$

$$f(x) = \frac{|2x|}{x} = \begin{cases} \frac{2x}{x}, & x>0 \\ \frac{2x}{-x}, & x<0 \end{cases} = \begin{cases} 2, & x>0 \\ -2, & x<0 \end{cases}$$

$$\lim_{x\to 0-} f(x) = -2; \quad \lim_{x\to 0+} f(x) = 2$$

The one-sided limits do not match $\Rightarrow$ the limit as $x \to 0$ does not exist.

(b) $\lim_{x\to+\infty} \sqrt{x^2+x}-x$ $= "+\infty - (+\infty)" \Rightarrow$ needs work.

$$f(x) = \sqrt{x^2+x}-x = \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}$$

$$= \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x} = \frac{1}{\sqrt{1+\frac{1}{x}}+1}$$

$$\xrightarrow[x\to+\infty]{} \frac{1}{1+1} = \frac{1}{2}$$

(c) $\lim_{x\to-\infty} \frac{100e^x}{e^x+e^{-5}}$.

$$e^x \xrightarrow[x\to-\infty]{} 0 \Rightarrow \frac{100e^x}{e^x+e^{-5}} \longrightarrow \frac{100\cdot 0}{0+e^{-5}} = 0$$

(d) (bonus: 2 points) In the last example, how large and negative does x need to be so that $f(x) = \frac{100e^x}{e^x+e^{-5}}$ is within 0.01 from its limit as $x \to -\infty$?

$$\frac{100e^x}{e^x+e^{-5}} < 0.01 = 10^{-2}$$

$$\frac{e^x}{e^x+e^{-5}} < 10^{-4}$$

$$10^4 e^x < e^x + e^{-5}$$

$$(10^4-1)e^x < e^{-5}$$

$$e^x < \frac{e^{-5}}{10^4-1}$$

$$x < \ln \frac{e^{-5}}{10^4-1}$$

$$x \lesssim -14.21$$

3. (6 points) Given $f(x) = x^2 - 0.5x$.

(a) Find the derivative $f'(x)$ by definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 0.5(x+h) - x^2 + 0.5x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{0.5x} - 0.5h - \cancel{x^2} + \cancel{0.5x}}{h}$$

$$= \lim_{h \to 0} 2x + h - 0.5 = 2x - 0.5$$

(b) Find the average rate of change of f over the interval $[0, 3]$.

$$ARC = \frac{f(3) - f(0)}{3 - 0} = \frac{(3^2 - 0.5 \cdot 3) - 0}{3}$$

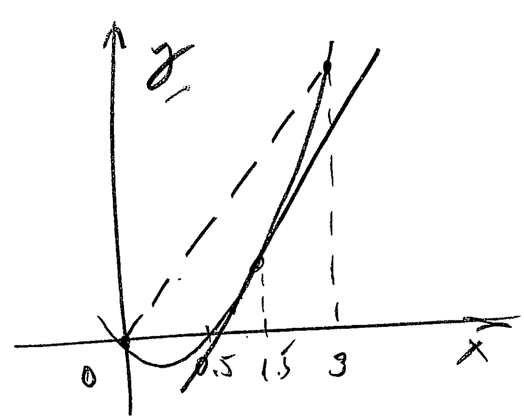
$$= \frac{7.5}{3} = 2.5$$

(c) Find the value c in the interval $(0, 3)$ at which the instantaneous rate of change of f is equal to the average rate of change over $[0, 3]$. Illustrate by a graph.

$$2x - 0.5 = 2.5$$

$$2x = 3$$

$$x = \frac{3}{2} = 1.5$$



The tangent line at $x = 1.5$ is parallel to the secant line through $(0,0)$ and $(3, 7.5)$.

Continued...

4. (6 points) An enviromental study suggests that the level of NO_2 pollution in the air is modelled by the function

$$P(t) = 78.2 + 3.2t - 0.04t^2 \quad [\text{ parts per billion (ppb) }]$$

t days after the start of observation.

(a) Find the derivative $\frac{dP}{dt}$.

$$\frac{dP}{dt} = 3.2 - 0.04 \cdot 2t = 3.2 - 0.08t$$

$$\left[\frac{\text{ppb}}{\text{day}}\right]$$

(b) Find the value $\left.\frac{dP}{dt}\right|_{t=45}$, specify units, and write a sentence to describe the meaning of the obtained value.

$$\left.\frac{dP}{dt}\right|_{t=45} = 3.2 - 0.08 \cdot 45$$

$$= 3.2 - 3.6 = -0.4 \quad \left[\frac{\text{ppb}}{\text{day}}\right]$$

After 45 days the level of NO is decreasing at a rate of 0.4 ppb per day.

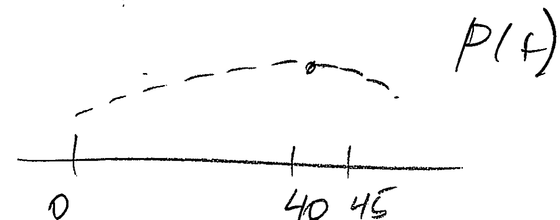
(c) When will the level of pollution start to decline?

$$\frac{dP}{dt} = 0$$

$$3.2 - 0.08t = 0$$

$$t = \frac{3.2}{0.08}$$

$$t = 40 \quad [\text{days}]$$



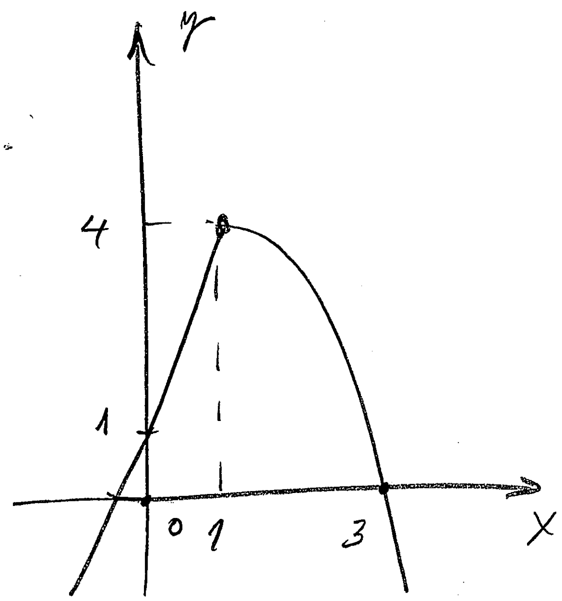
(P(t) is a parabola facing downward.)

Continued...

5. (6 points) (a) Determine the values that need to be assigned to k and to $f(1)$ so that the function $f(x)$ would be continuous at $x = 1$:

$$f(x) = \begin{cases} 1 + kx, & x < 1 \\ 4 - (x-1)^2, & x > 1. \end{cases}$$

Sketch a graph of f for this value k.



$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 1 + kx$$

$$= 1 + k$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 4 - (x-1)^2$$

$$= 4 - 0 = 4$$

$$1 + k = 4$$
$$k = 4 - 1$$
$$k = 3$$

$$f(1) = 4$$

(b) Is the function f obtained in part (a) differentiable at $x = 1$? Write a sentence to justify your answer.

No. Differentiable means there is a well-defined tangent line; here the slope of the graph on the left $= 3$, while the slope of the graph on the right is 0.

6. (6 points) Find the derivative of the function $f(x)$ and use it to determine on which intervals is the function increasing or decreasing:

$$f(x) = 50 + 20x - e^x.$$

$$f'(x) = 0 + 20 - e^x$$
$$= 20 - e^x.$$

incr: $f'(x) > 0$

decr: $f'(x) < 0$

First solve $f'(x) = 0$:

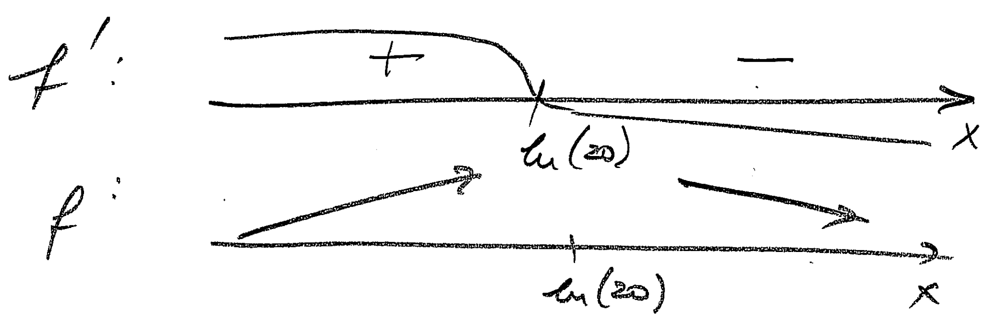
$$20 - e^x = 0$$
$$20 = e^x$$
$$e^x = 20$$
$$x = \ln(20) \approx 2.9957$$

If $x < \ln(20)$ then $e^x < 20$

$\Rightarrow f'(x) = 20 - e^x > 0$

If $x > \ln(20)$ then $e^x > 20$

$\Rightarrow f'(x) = 20 - e^x < 0$



incr: $(-\infty, \ln(20))$

decr: $(\ln(20), \infty)$.

Continued...

7. (6 points) Use the Intermediate Value Theorem to prove that the following equation has at least one solution:

$$x^3 - 2x^2 - 8x = 4.$$

$f(x) = x^3 - 2x^2 - 8x - 4$

$f(0) = -4$

$f(10) = 1000 - 200 - 80 - 4 =$

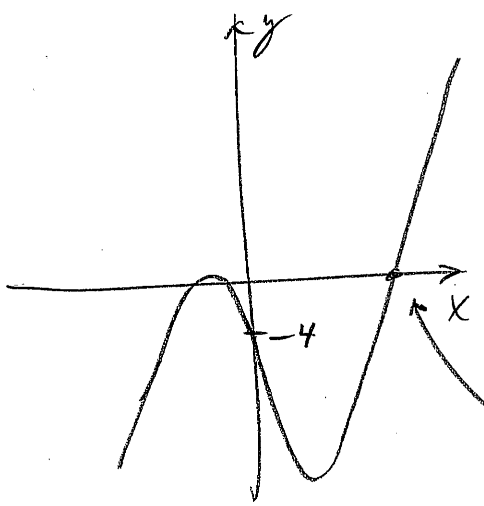
$= 1000 - 284 = 712 > 0$

$f(x)$ is continuous on $[0, 10]$

Therefore, by IVT there must be a value c such that $f(c) = 0$

$\Rightarrow c^3 - 2c^2 - 8c = 4.$

(b) Find the largest positive solution of the equation in part (a) accurate to three decimal places. Show your work!



Calculator:

$Y = X^3 - 2X^2 - 8X - 4$

Graph

[CALC] → zero

returns

$\begin{cases} X = 4.1563252 \\ Y = 1E{-12} \end{cases}$

$X \simeq 4.156$

Table of formulas

Average Rate of Change over $[a, b]$: $\text{ARC} = \dfrac{f(b) - f(a)}{b - a}$.

Instantaneous Rate of Change at a: $\text{IRC} = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$.

Derivative at x: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Derivative Notations:

$$f'(x) = \frac{dy}{dx}; \quad f'(a) = f'(x)\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a}$$

Derivative Rules:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$f_1(x) \pm f_2(x)$	$f_1'(x) \pm f_2'(x)$
$cf_1(x)$	$cf_1'(x)$
e^{mx}	me^{mx}
b^x	$(\ln b)b^x$

Intermediate Value Theorem: If $f(x)$ is continuous on $[a, b]$, and L is a value strictly between $f(a)$ and $f(b)$ then for some c in (a, b) we must have $f(c) = L$.

Mean Value Theorem: If $f(x)$ is differentiable on (a, b), continuous on $[a, b]$ then for some c in (a, b) we must have

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The end.