

Name: (print) _____

Student No. : Solutions.

This test paper has 7 pages. The duration of the test is 50 minutes. There are 7 questions in the main part (62 points) and one bonus question (6 points).

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Note: Write solutions in the space provided. If out of space you may use the back of the page.

Important: All answers should be justified. Show your work clearly and completely, explaining your steps.

1. (6 points) Let $f(x) = 1 + x^2$ and $g(x) = \ln x$. Find

(a) $f(g(x))$ and its domain

$$f(g(x)) = 1 + (\ln x)^2$$

Domain: $x > 0$ (for $\ln x$ to be defined)

(b) $g(f(x))$ and its domain

$$g(f(x)) = \ln(1 + x^2)$$

Domain: all real x

($1 + x^2 > 0$ always)

2. (10 points) 25 g of polonium 210 is decaying exponentially. After 45 days 20 g of polonium 210 is left.

(a) Write an exponential equation to express the amount of polonium as a function of time.

$$A(t) = A_0 e^{kt}$$

$$A(0) = A_0 e^0 = A_0 = 25$$

$$A(45) = A_0 e^{45k} = 25e^{45k} = 20$$

$$e^{45k} = \frac{20}{25} = 0.8$$

$$45k = \ln 0.8$$

$$k = \frac{\ln 0.8}{45} = -0.00496$$

(b) Find the half-life of polonium 210.

$$\text{Half life: } \frac{1}{2} = 1 \cdot e^{kt}$$

$$\ln \frac{1}{2} = kt$$

$$t = \frac{\ln \frac{1}{2}}{k} = 45 \frac{\ln \frac{1}{2}}{\ln 0.8}$$

$$= 139.78 \text{ days.}$$

3. (12 points) Let $f(x) = 1 - \frac{x^2}{2}$.

(a) Use the definition of the derivative to find $f'(x)$.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1 - \frac{(x+h)^2}{2} - \left(1 - \frac{x^2}{2}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-x^2 - 2xh - h^2 + x^2}{2h} \\
 &= \lim_{h \rightarrow 0} \left(-x - \frac{h}{2}\right) = -x.
 \end{aligned}$$

(b) Find the equation of the tangent line to the graph at $x = -2$.

$$y - y_0 = m(x - x_0)$$

$$x_0 = -2$$

$$y_0 = f(-2) = 1 - \frac{(-2)^2}{2} = -1$$

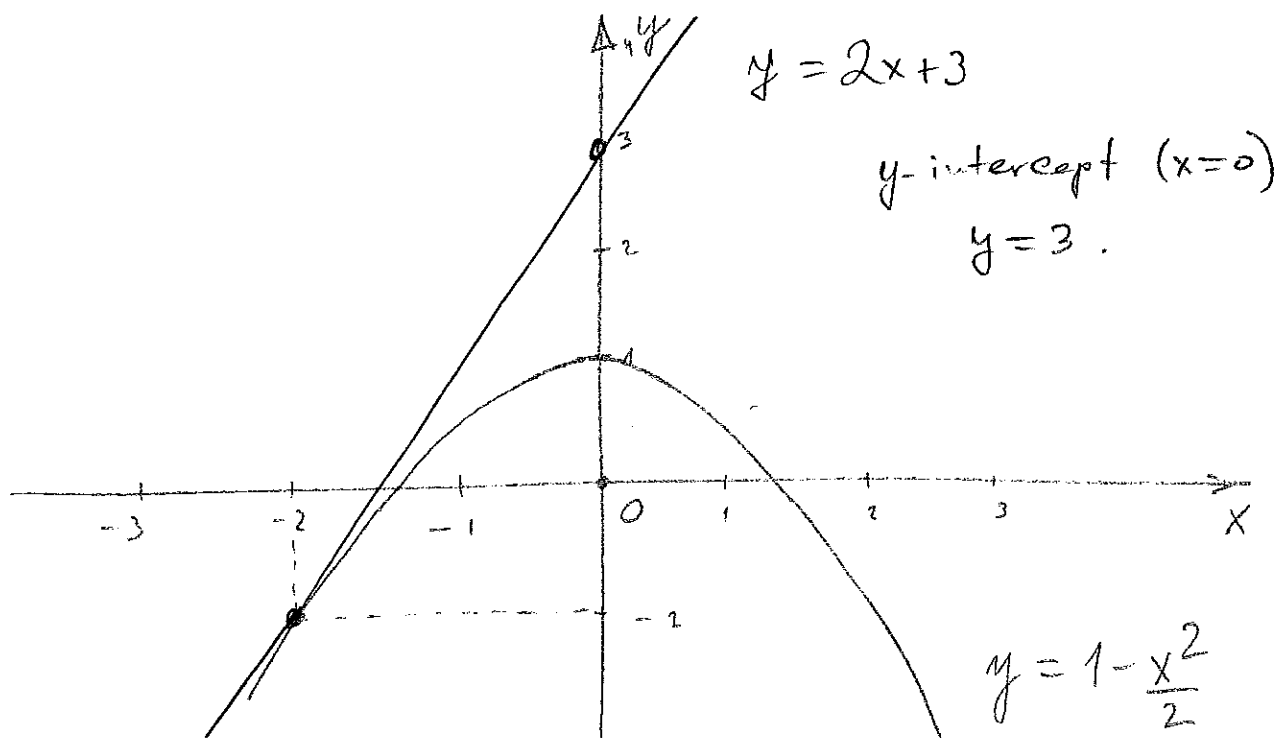
$$m = f'(-2) = -(-2) = 2$$

$$y + 1 = 2(x + 2)$$

$$y + 1 = 2x + 4$$

$$y = 2x + 3.$$

(c) Sketch the tangent line from part (b) on the graph below. Find the y -intercept.



4. (10 points) Find the limits

$$\begin{aligned}
 \text{(a)} \quad \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2} &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} \\
 &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1})^2 - 2^2} \\
 &= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{x+1-4} \\
 &= \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(\sqrt{x+1}+2)}{\cancel{x-3}} \\
 &= \lim_{x \rightarrow 3} (\sqrt{x+1}+2) = 4
 \end{aligned}$$

$\swarrow$
 2

Continued...

(8 points)

6. Let $f(x) = 2\cos(\pi x) - 1$.

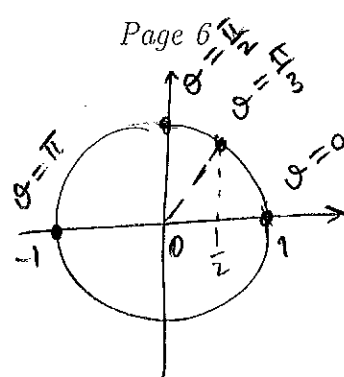
(a) Find $f(0)$, $f(\frac{1}{3})$, $f(\frac{1}{2})$, $f(1)$.

$$f(0) = 2\cos 0 - 1 = 2 - 1 = 1$$

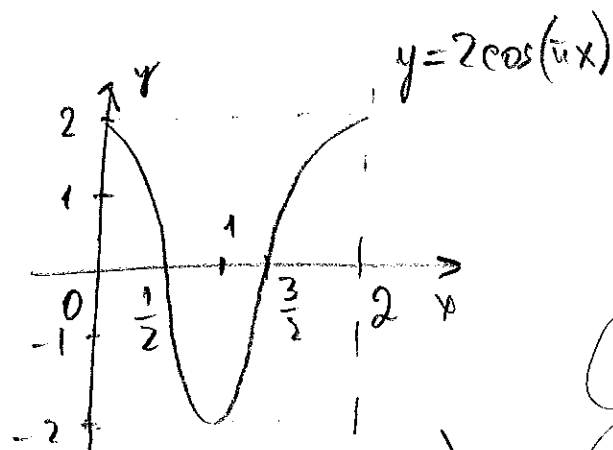
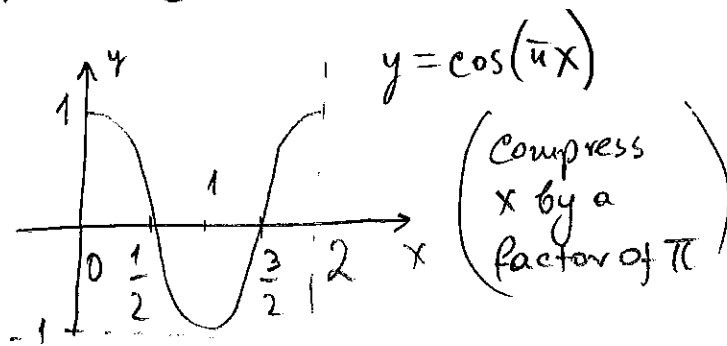
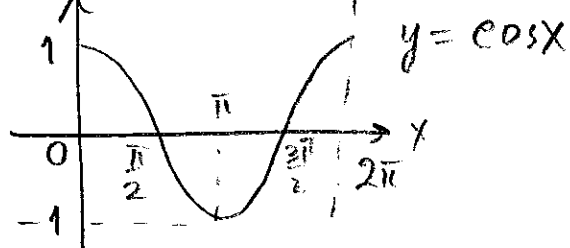
$$f(\frac{1}{3}) = 2\cos \frac{\pi}{3} - 1 = 2 \cdot \frac{1}{2} - 1 = 0$$

$$f(\frac{1}{2}) = 2\cos \frac{\pi}{2} - 1 = 0 - 1 = -1$$

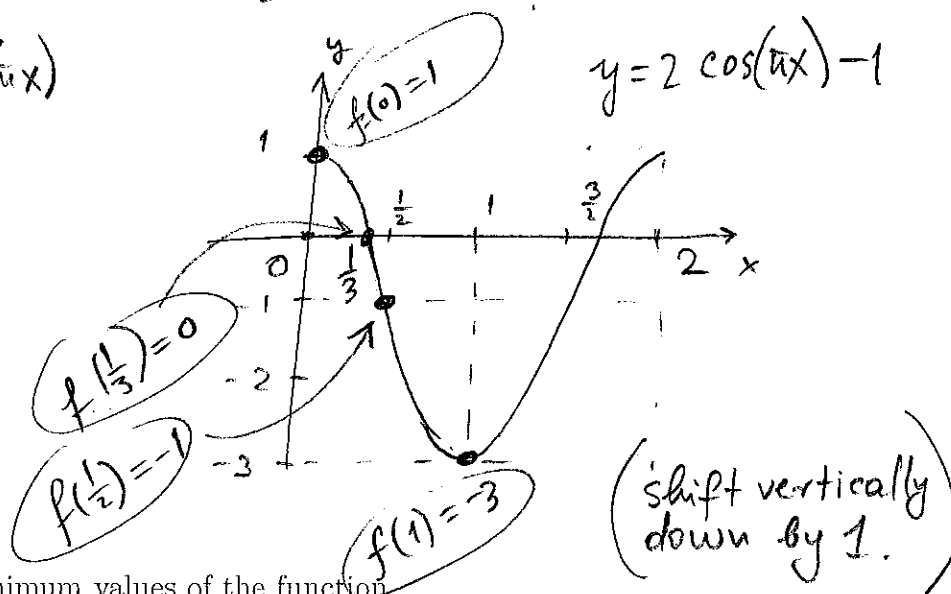
$$f(1) = 2\cos \pi - 1 = 2 \cdot (-1) - 1 = -3$$



(b) Graph $f(x)$ over an interval of one period.



(stretch y by a factor of 2)



(c) Find the maximum and minimum values of the function.

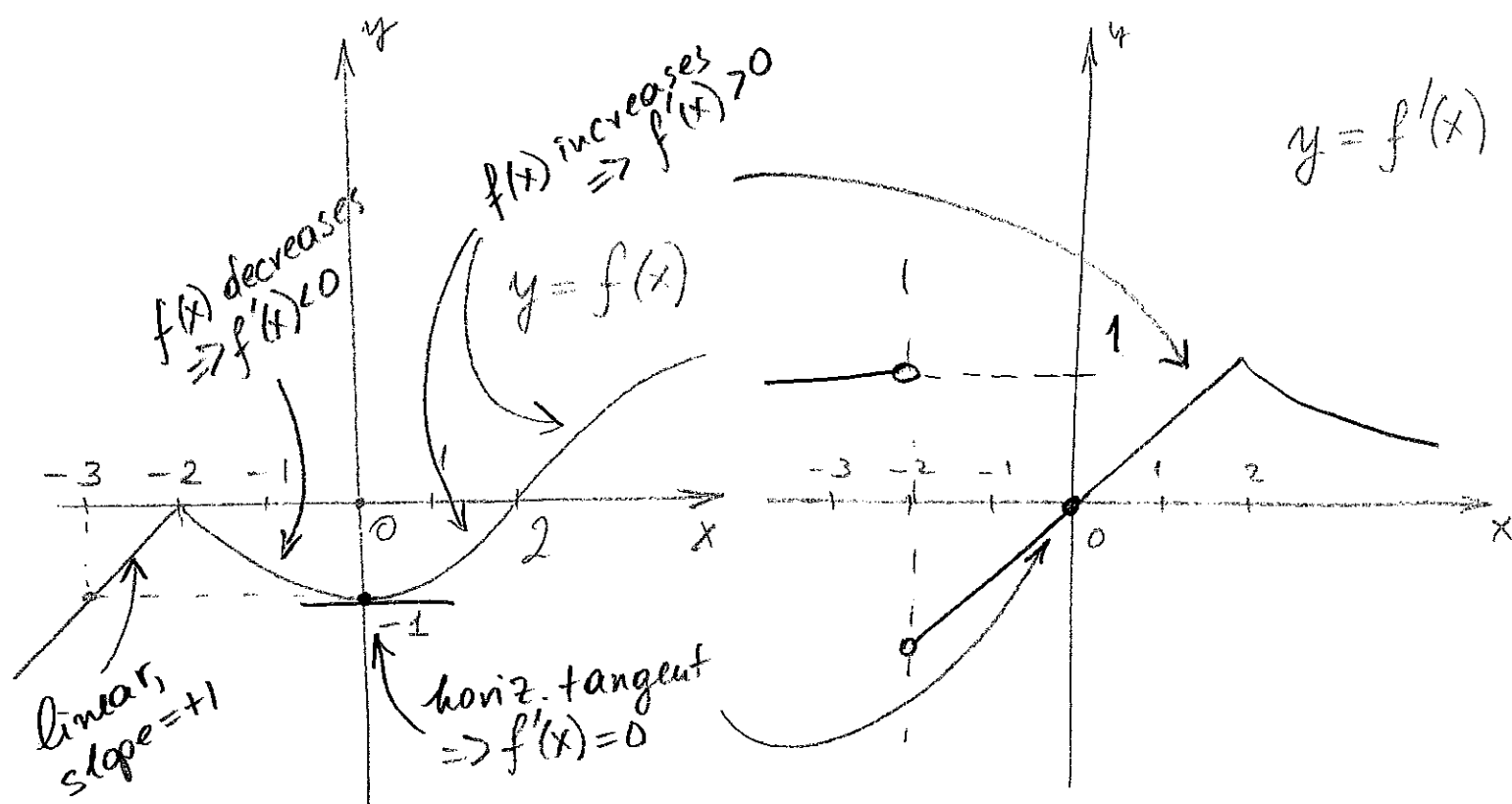
From the graph (also confirmed by part (a))

$$\text{maximum} = 1$$

$$\text{minimum} = -3$$

(8 points)

7. Sketch the graph of the derivative for the function shown.

8. (bonus: 6 points) Solve the equation, $\ln x - \ln(x-2) = \ln \frac{1}{2}$.

$$\ln \frac{x}{x-2} = \ln \frac{1}{2}$$

$$\frac{x}{x-2} = \frac{1}{2}$$

$$2x = x-2$$

$$x = -2$$

However, this is not a solution since
cannot plug it back in:

$\ln(-2)$ not defined.

$\Rightarrow$ Equation has no solution.

The end.