Wed., Oct. 24, 2007, MATH 255A Midterm Test 2

Name: (print) ____________________

Student ID No. : Solutions.

This test paper has 6 pages. The duration of the test is 50 minutes. There are 5 questions in the main part (51 points) and one bonus question (6 points).

Your scores: (do not enter answers here)

1	2	3	4	5	6	total

Note: Write solutions in the space provided. If out of space you may use the back of the page. *Important: All answers should be justified. Show your work clearly and completely, explaining your steps.*

1. (6 points) If $f(1) = 5$, $g(1) = 3$, $f'(1) = 0$, $g'(1) = 2$, find $h'(1)$, where

$$h(x) = (f(x))^2\, g(x)\,.$$

$$h'(x) = 2f(x)f'(x)g(x) + (f(x))^2 g'(x)$$

$$h'(1) = 2f(1)\underbrace{f'(1)}_{=0}g(1) + (f(1))^2 g'(1)$$

$$= 0 + 5^2 \cdot 2 = 25 \cdot 2 = 50.$$

2. (10 points) A particle moves along a straight line. The distance of the particle from the origin is given by

$$s(t) = 50 + 6\sin(2\pi t)$$

(a) Find the velocity $v(t) = s'(t)$ at $t = 0$, $t = \frac{1}{3}$, $t = 1$.

$$v(t) = 6 \cdot 2\pi \cdot \cos(2\pi t) = 12\pi \cos(2\pi t)$$

$$v(0) = 12\pi \cos(0) = 12\pi.$$

$$v(\tfrac{1}{3}) = 12\pi \cos\left(\tfrac{2\pi}{3}\right) = -6\pi.$$

$$v(1) = 12\pi \cdot \cos(2\pi) = 12\pi.$$

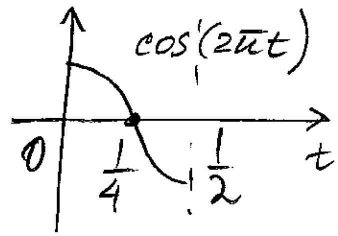
(b) For which t in the interval $0 \leq t \leq \frac{1}{2}$ is the particle at rest (the velocity $v(t)$ equals zero)?

$$12\pi \cos(2\pi t) = 0$$

$$\cos(2\pi t) = 0$$

$$2\pi t = \frac{\pi}{2}$$

$$t = \frac{1}{4}.$$



(c) Find the maximal and minimal distance of the particle from the origin.

$$\max s(t) = 50 + 6 \cdot 1 = 56.$$

$$\min s(t) = 50 + 6(-1) = 44.$$

3. (10 points) The function $h(z)$ is given by

$$h(z) = (\sqrt{z} - 1)^3 + (\sqrt{z} - 1)^2 + 1.$$

(a) Write $h(z)$ as a composition of two functions, $f(g(z))$. (There may be more than one way to do this.)

$$f(u) = u^3 + u^2 + 1$$

$$g(z) = \sqrt{z} - 1$$

Then $h(z) = f(g(z))$.

$$f'(u) = 3u^2 + 2u$$

$$u = g'(z) = \frac{1}{2\sqrt{z}}$$

(b) Find $h'(1)$.

$$h'(z) = f'(u)\, g'(z) = (3u^2 + 2u)\frac{1}{2\sqrt{z}}$$

$$= \frac{1}{2\sqrt{z}}\left(3(\sqrt{z}-1)^2 + 2(\sqrt{z}-1)\right)$$

$$h'(1) = \frac{1}{2} \cdot (0 + 0) = 0.$$

4. (10 points) The concentration of pollutants in a river, x miles downstream from a polluting paper mill, is given by the function

$$p(x) = 0.14\,e^{-4x} \qquad \text{(in grams per liter)}.$$

(a) Find the rate of change of the concentration with respect to the distance x, for $x = 1$ and $x = 2$ miles.

$$p'(x) = 0.14\,(-4)\,e^{-4x} = -0.56\,e^{-4x}$$

$$p'(1) = -0.56\,e^{-4}$$

$$p'(2) = -0.56\,e^{-8}$$

(b) Show that the rate of change of the concentration at x is proportional to $p(x)$. Find the proportionality constant.

$$\frac{p'(x)}{p(x)} = \frac{-0.56\,e^{-4x}}{0.14\,e^{-4x}} = -4$$

$$p'(x) = k\,p(x), \quad k = -4.$$

(c) What happens to the rate of change as x increases?

$p'(x)$ decreases in magnitude and goes to zero.

$$\lim_{x \to \infty} p'(x) = \lim_{x \to \infty} 0.14\,\underbrace{e^{-4x}}_{\to 0} = 0.$$

Continued...

5. (15 points) Find the derivatives of the functions

(a) $f(t) = \dfrac{2}{\sqrt{4-t^2}} = 2\left(4-t^2\right)^{-\frac{1}{2}}$

$$f'(t) = 2\cdot\left(-\frac{1}{2}\right)\left(4-t^2\right)^{-\frac{3}{2}} = -\left(4-t^2\right)^{-\frac{3}{2}} = -\frac{1}{(4-t^2)^{3/2}}.$$

(b) $g(z) = (e^{2z} + \ln|7z|)^5$

$$g'(z) = 5\left(e^{2z} + \ln|7z|\right)^4\left(2e^{2z} + \frac{1}{z}\right)$$

(c) $r(x) = \tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$

$$r'(x) = \left(\frac{\sin(2x)}{\cos(2x)}\right)' = \frac{2\cos(2x)\cdot\cos(2x) - \sin(2x)(-2\sin(2x))}{\cos^2(2x)}$$

$$= \frac{2\left(\cos^2(2x) + \sin^2(2x)\right)}{\cos^2(2x)}$$

$$= \frac{2}{\cos^2(2x)}.$$

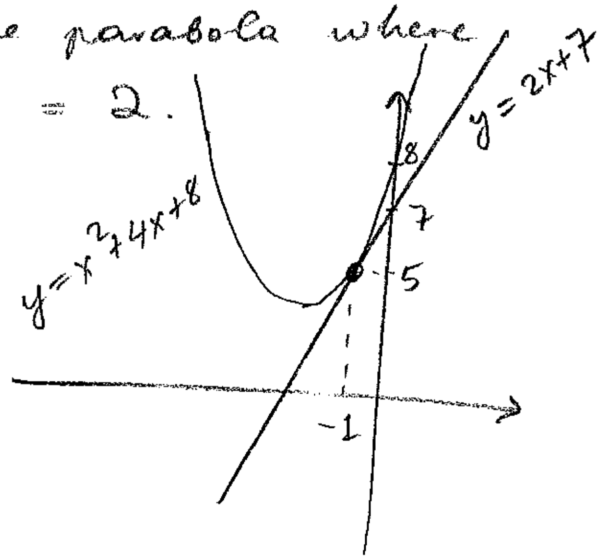
6. (bonus: 6 points) The line $y = 2x + 7$ is tangent to the parabola

$$y = x^2 + 4x + 8$$

at a certain point (x_0, y_0). Find this point.

The line $y = 2x + 7$ has slope $= 2$.

Find the point on the parabola where the slope of the tangent $= 2$.



$f(x) = x^2 + 4x + 8$

$f'(x) = 2x + 4$

$2x + 4 = 2$

$2x = -2$

$x = -1.$

$f(-1) = 1 - 4 + 8 = 5$

The point (x_0, y_0) is $(-1, 5)$

(Check that the line $y = 2x + 7$ goes through $(-1, 5)$:
$5 = 2\cdot(-1) + 7$)

The end.