Higher Derivatives and Approximations. (3.6)

$$f(x) \xrightarrow[\left(\frac{d}{dx}\right)]{\text{take derivative}} f'(x)$$

$$f'(x) \xrightarrow[\left(\frac{d}{dx}\right)]{} f''(x) \quad - \text{ second derivative.}$$

Def: the n-th derivative of a function $f(x)$ is defined recursively as

$$f^{(n)}(x) = \lim_{h\to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$$

(so n-th derivative is the (first) derivative of $(n-1)$-th derivative.)

Example: (1) $f(x) = x^3 + 3x^2 + 6x + 5$

$f'(x) = 3x^2 + 6x + 6$

$f''(x) = 6x + 6$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 4$.

Notation:
f' – first
f'' – second
f''' – third
$f^{(4)}$ – fourth
(for derivative of order 4 and higher.)

(2)

$f(x) = \sin(2x)$

$f' = 2\cos(2x)$

$f'' = -4\sin(2x)$

$f''' = -8\cos(2x)$

$f^{(4)} = 16\sin(2x)$

$\vdots$

$f^{(101)} = 2^{101}\cos(2x)$

(since $f^{(100)} = 2^{100}\sin(2x)$)

(3) $f(x) = xe^x$

$f' = e^x + xe^x = (x+1)e^x$

$f'' = e^x + (x+1)e^x = (x+2)e^x$

$f''' = e^x + (x+2)e^x = (x+3)e^x$

$\vdots$

$f^{(n)} = (x+n)e^x$

for any $n = 0, 1, 2, \ldots$

($n=0$ is the original function.)

Geometric Significance of the Second Derivative

The first derivative determines the slope of the tangent line (increase/decrease of the function)

The second derivative determines whether the slope of the graph (the tangent line) is increasing or decreasing.

For an interval $I = (a, b)$:

* If $f''(x) > 0$ on I then $f'(x)$ is increasing on I

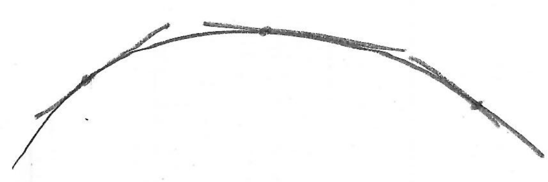
Tangent lines below the graph of f **

A function f with such property is called

Concave up

* If $f''(x) < 0$ on I then $f'(x)$ is decreasing on I

tangent lines above the graph of f **



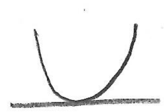
A function f with such property is called

concave down

Ex: $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2 > 0$

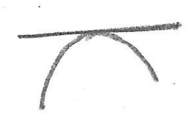


concave up

$f(x) = -x^2$

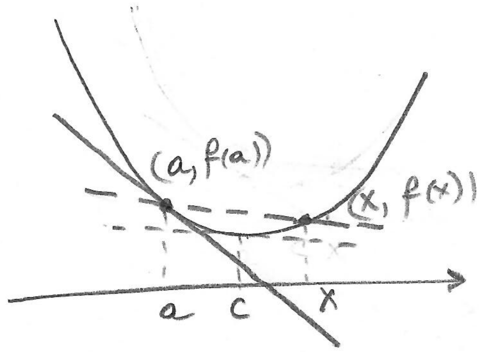
$f'(x) = -2x$

$f''(x) = -2 < 0$



concave down.

** Footnote:



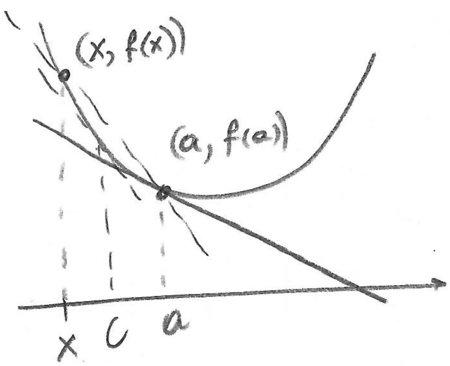
suppose $f'(x)$ is incr on I,
$x = a$ is a point in I

Then for $x > a$

(MVT) $$\frac{f(x)-f(a)}{x-a} = f'(c) > f'(a)$$

$$f(x) - f(a) > f'(a)(x-a)$$

$$\Rightarrow f(x) > \underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}}$$



For $x < a$,

(MVT) $$\frac{f(x)-f(a)}{x-a} = f'(c) < f'(a)$$

because c is between x and a

$\Rightarrow$ $(x-a<0)$ $$f(x) - f(a) > f'(a)(x-a)$$

$$f(x) > \underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}}$$

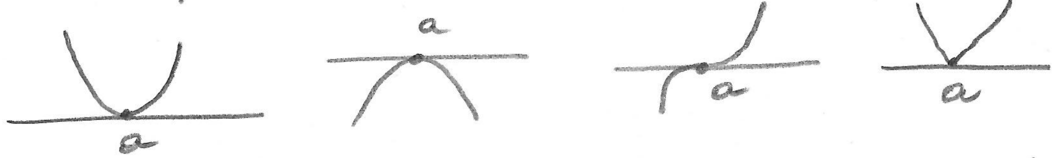
In both cases $f(x) > f(a) + f'(a)(x-a)$

$\Rightarrow$ graph is above the tangent line.

Similar argument shows that
when $f'(x)$ is decreasing on I,
then the graph of $f(x)$ is below any tangent line.

Def: A value $x = a$ (or a point $(a, f(a))$) is a critical point for a function $f(x)$ if $f'(a) = 0$ or $f'(a)$ D.N.E. (but $f(a)$ exists and $f(x)$ is continuous at a.)

Ex: Critical points



Def: Maximum = greatest value of the function
Minimum = least value of the function

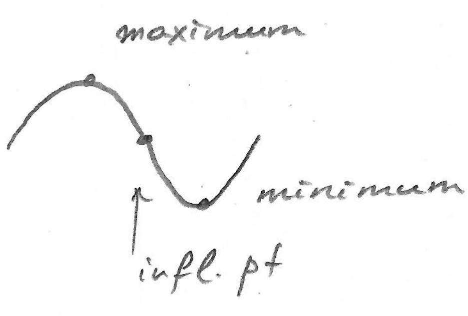
As the above examples indicate, critical points frequently correspond to maxima and minima of functions (but they do not have to.)

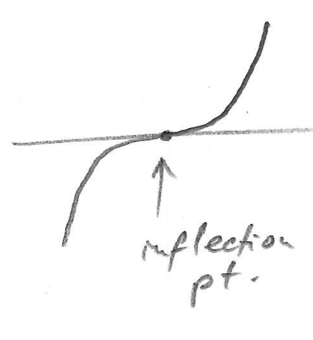
Global maximum/minimum: applies to the whole domain of the function

Local maximum/minimum: applies to a small interval about $x = a$.

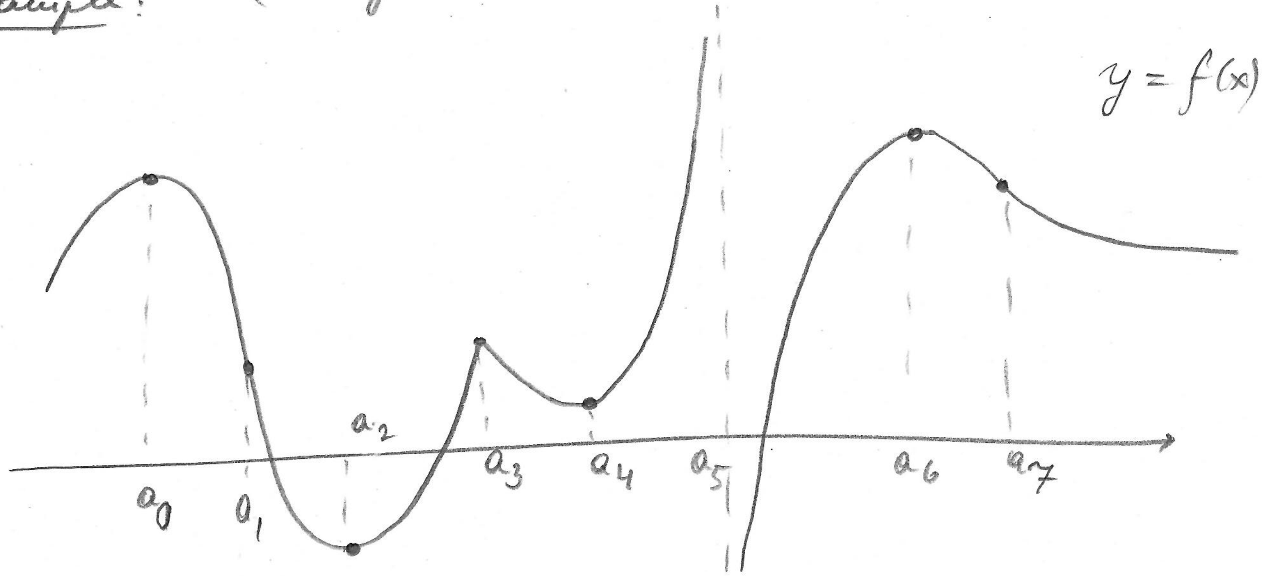
Def: A value $x = a$ (or a point $(a, f(a))$) is an inflection point, if it separates an interval on which f is concave down from an interval on which f is concave up.
(The graph of f changes concavity at an inflection point.)

Ex:





Example: (Graphical)



a_0 - critical pt
$f(a_0)$ - local maximum
a_2 - critical pt
$f(a_2)$ - local minimum
a_3 - critical point; also inflection point
$f(a_3)$ - local maximum
a_4 - crit. pt
$f(a_4)$ - local min
a_5 - vertical asymptote
($f(a_5)$ undefined)
a_6 - crit. pt;
$f(a_6)$ - local max.
a_7 - inflection point

a_1 - inflection point
- graph changes from concave down to concave up

Example (Computational)

$$f(x) = x e^{-x}$$

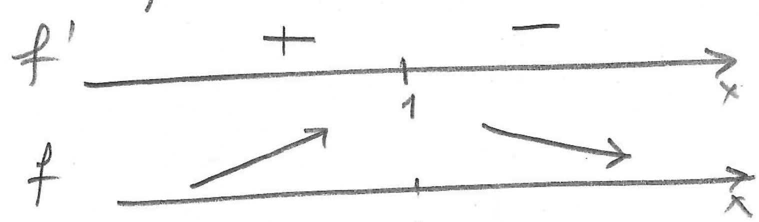
Compute $f'(x)$, $f''(x)$, find out where $f(x)$ is
(i) increasing
(ii) decreasing
(iii) concave up
(iv) concave down

Sketch a graph.

$$f'(x) = e^{-x} + x \cdot e^{-x} \cdot (-1) = (1-x)e^{-x}$$

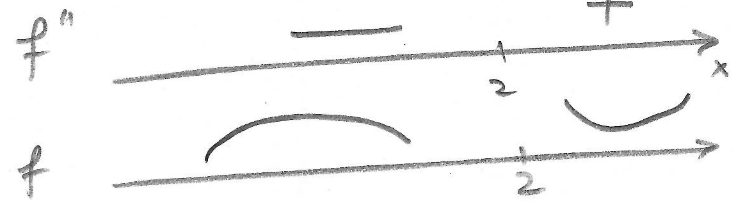
$f'(x) > 0$ when $1-x>0 \Rightarrow x<1$
$f'(x) < 0$ when $1-x<0 \Rightarrow x>1$

[sign diagram] →



$$f''(x) = -e^{-x} + (1-x)\cdot e^{-x}(-1) = (x-2)e^{-x}$$

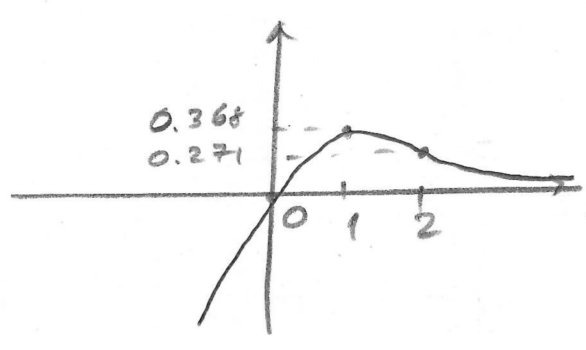
$f''(x) > 0$ when $x-2>0 \Rightarrow x>2$
$f''(x) < 0$ when $x-2<0 \Rightarrow x<2$



Compute a few values:

$f(0) = 0$ (intercept)
$f(1) = e^{-1} \approx 0.368$
$f(2) = 2e^{-2} \approx 0.271$

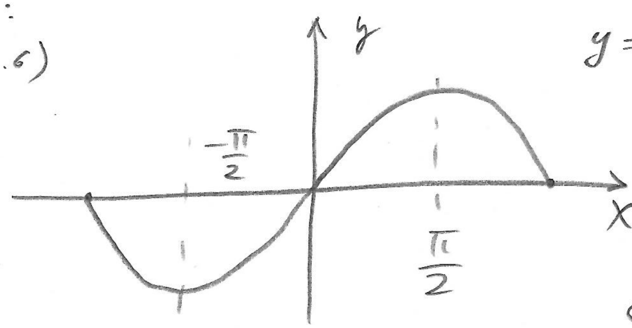
Graph:



Note:

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0.$$

Ex:
(Ex 2, 3.6)



$y = \sin x$; on $(-\pi, \pi)$.

incr: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

decr: $\left(-\pi, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$

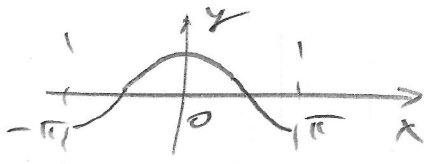
Concave up: $(-\pi, 0)$

concave down: $(0, \pi)$.

Verify with derivatives:

First deriv: $(\sin x)' = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

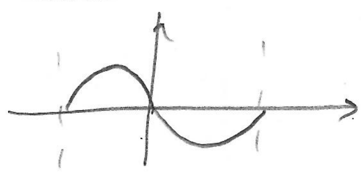
< 0 on $\left(-\pi, -\frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$



$(\sin x)'' =$

Second deriv: $= (\cos x)' = -\sin x > 0$ on $(-\pi, 0)$

< 0 on $(0, \pi)$.

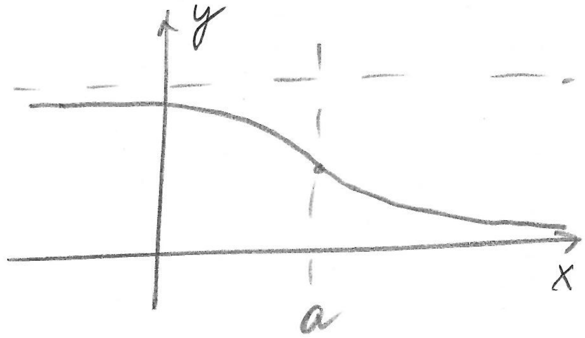


$x = \pm\frac{\pi}{2}$ - critical points; $(f'(x) = 0)$,

$x = 0$ — point of inflection $(f''(x) = 0)$

Ex Sigmoidal function

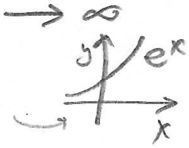
(Ex 3, 3.6)



$$f = \frac{1}{1 + e^{x-a}}$$

As $x \to \infty$ $x - a \to \infty$

$e^{x-a} \to \infty$



$\frac{1}{1 + e^{x-a}} \to 0$

As $x \to -\infty$ $x - a \to -\infty$

$e^{x-a} \to 0$

$\frac{1}{1 + e^{x-a}} \to 1$.

$$f' = \left(\frac{1}{1+e^{x-a}}\right)' = \frac{-e^{x-a}}{(1+e^{x-a})^2} < 0$$

f is decreasing on $(-\infty, \infty)$

$$f'' = \left(\frac{-e^{x-a}}{(1+e^{x-a})^2}\right)' = -\frac{e^{x-a}(1+e^{x-a})^2 - e^{x-a} \cdot \underbrace{2(1+e^{x-a})}_{\cdot e^{x-a}}}{(1+e^{x-a})^4}$$

$$= -\frac{e^{x-a}(1+e^{x-a})(1+e^{x-a}-2e^{x-a})}{(1+e^{x-a})^4}$$

$$= \frac{e^{x-a}(e^{x-a}-1)}{(1+e^{x-a})^3}$$

$f'' = 0$ when $e^{x-a} - 1 = 0$

$$e^{x-a} = 1$$

$$x - a = 0$$

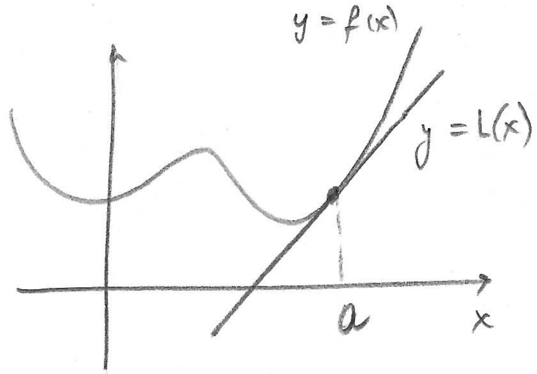
$$x = a$$

$x = a$ – point of inflection

Quadratic (Second Order) Approximations

Recall: Linear Approximation:

$$L(x) = f(a) + \underbrace{f'(a)(x-a)}_{\text{tangent line at } a}$$



Want to go one step further, and choose a parabola $Q(x) = A + B(x-a) + C(x-a)^2$ such that it matches the function $f(x)$ better than a straight line.

Choose $Q(x)$ so that

* $f(a) = Q(a)$
* $f'(a) = Q'(a)$
* $f''(a) = Q''(a)$

The function value and the first two derivatives match.

$f(a) = Q(a) \Rightarrow f(a) = A$ (A is determined)

$Q'(x) = B + 2C(x-a) \Rightarrow f'(a) = B$ (B is determined)

$Q''(x) = 2C \Rightarrow f''(a) = 2C \Rightarrow C = \frac{f''(a)}{2}$ (C is determined)

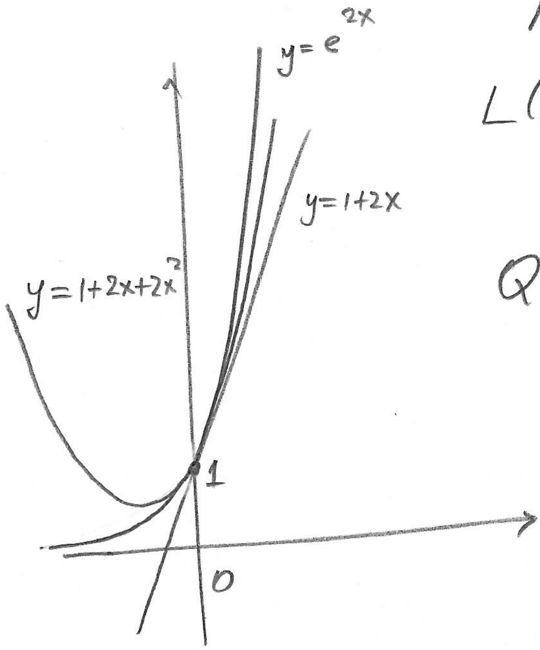
$$Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$f(x) \approx Q(x)$ — quadratic (second order) approximation

Example: $f(x) = e^{2x}$, $a = 0$ (as in Ex. 7, p. 257.)

$$f'(x) = 2e^{2x}, \quad f'(a) = f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(a) = f''(0) = 4$$



$$L(x) = f(a) + f'(a)(x-a)$$

$$= 1 + 2(x-a) = 1+2x$$

$$Q(x) = 1 + 2(x-a) + \frac{1}{2}\cdot 4(x-a)^2$$

$$= 1+2x+2x^2$$

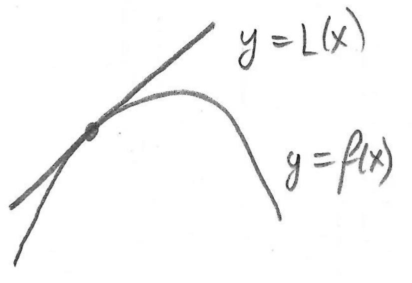
$Q(x)$ is better match for the fuction if we zoom in onto $(a, f(a))$

The quadratic approximation picks up the shape of the graph near $x=a$:

if $\underline{f''(a) > 0}$ then $f(x)$ is concave up near $x=a$

$\Rightarrow$ $L(x)$ is an <u>underestimate</u> of $f(x)$ (as in the figure above.)

if $\underline{f''(a) < 0}$ then $f(x)$ is concave down near $x=a$



$\Rightarrow$ $L(x)$ is an overestimate of $f(x)$ near $x=a$.

(as on the figure on the left.)