Extra Credit Problem for the Final Exam

Show that the function

$$f(x,y) = \begin{cases} xy\,\dfrac{y^2 - x^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$, the second-order partial derivatives f_{xy} and f_{yx} exist at $(0,0)$, however

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

[Hint: to show that the partial derivatives f_x, f_y, f_{xy}, f_{yx} exist at $(0,0)$ you'll need to use the definition of the derivative as a difference quotient.] Remark: There is no contradiction to the theorem about the equality of mixed partials, since f_{xy} and f_{yx} are not continuous at $(0,0)$.