Mon, Nov 23, 2015 **MATH 250, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

This test includes 8 questions (total of 52 points), on 8 pages. The last question is a bonus; the perfect score is 46 points. The duration of the test is 75 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

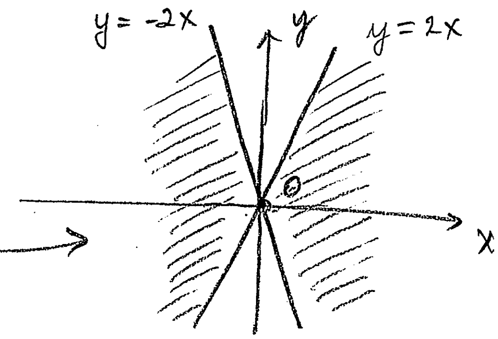
1. (6 points) Find the domain of the function and sketch a contour map (showing and labeling several level curves):

$$f(x, y) = \sqrt{4x^2 - y^2}.$$

$4x^2 - y^2 \geq 0$ for $\sqrt{\cdot}$ to be defined

$4x^2 \geq y^2$

$2|x| \geq |y|$



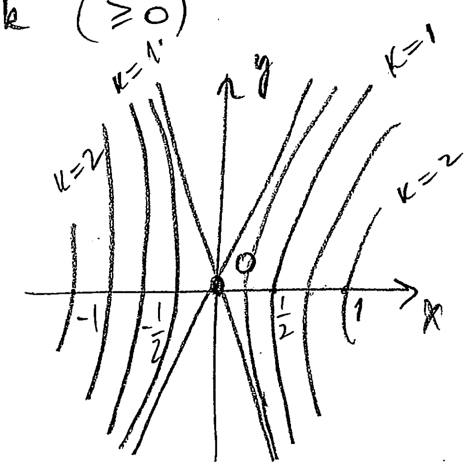
$D = \{ (x,y) : -2|x| \leq y \leq 2|x| \}$.

Level curves: $4x^2 - y^2 = k^2$ (≥ 0)

$k = 0$: pair of lines $2x = \pm y$

$k > 0$: hyperbolas

$$\frac{x^2}{(1/2k)^2} - \frac{y^2}{k^2} = 1$$



2. (6 points) (a) Determine whether the function $f(x, y)$ is continuous at $(0, 0)$:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

Try 2 different paths through the origin:

$y=0,\ x\to 0 \qquad f(x,0) = \frac{0}{x^2} = 0 \quad (x\neq 0)$

$y=x,\ x\to 0 \qquad f(x,x) = \frac{x^2}{x^2+2x^2} = \frac{1}{3} \quad (x\neq 0)$

Path limits are different

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.N.E.

$\Rightarrow f(x,y)$ is not continuous at $(0,0)$.

(b) Based on rules and theorems on limits and continuity justify the fact that the function from part (a) is continuous for $(x, y) \neq (0, 0)$.

$f(x,y)$ is a rational function $\left(\frac{\text{polynomial}}{\text{polynomial}}\right)$;

rational functions are continuous at every point of their domains.
(i.e. denominator $\neq 0$.)

$x^2+2y^2=0 \iff (x,y)=(0,0)$

$\Rightarrow f(x,y)$ is continuous at every point $(x_0,y_0)\neq(0,0)$.

Continued...

3. (6 points) The pressure, volume and temperature of <u>1 mole</u> of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals (kPa), V in liters (L), and T in Kelvins (K). Use differentials to to estimate the approximate change in V if P increases from 120 to 120.8 kPa and T decreases from 270 to 267 K.

$$PV = RT \qquad (R = 8.31)$$

$$V = \frac{RT}{P} ; \qquad \frac{\partial V}{\partial T} = \frac{R}{P} ; \qquad \frac{\partial V}{\partial P} = -\frac{RT}{P^2}$$

$$dV = \frac{\partial V}{\partial T} dT + \frac{\partial V}{\partial P} dP$$

$$= \frac{R}{P} dT - \frac{RT}{P^2} dP = \frac{R}{P}\left(dT - \frac{T}{P} dP\right)$$

when $T = 270$, $P = 120$, $dT = -3.0$, $dP = 0.8$

$$\Delta V \approx dV = \frac{8.31}{120}\left(-3.0 - \frac{270}{120} 0.8\right)$$

$$= -0.3324$$

4. (8 points) Find all critical points of the function

$$f(x, y) = x^3 + 3xy^2 - 6y^2 - 6x^2 + 2$$

and determine their type (local maximum/minimum, or saddle point(s)).

$$\begin{cases} f_x = 3x^2 + 3y^2 - 12x = 0 \\ f_y = 6xy - 12y = 0 \end{cases} \Rightarrow 6y(x-2) = 0$$

$\Rightarrow \quad y = 0 \quad$ or $\quad x = 2$

$y = 0 \Rightarrow \quad 3x^2 - 12x = 0 \Rightarrow 3x(x-4) = 0$
$\Rightarrow x = 0$ or $x = 4$

$x = 2 \Rightarrow \quad 3y^2 - 12 = 0 \Rightarrow y^2 = 4$
$\Rightarrow y = \pm 2$

Critical points: $(0,0)$, $(4,0)$
$(2,2)$, $(2,-2)$

$$D^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x-12 & 6y \\ 6y & 6x-12 \end{pmatrix}$$

$D^2 f(0,0) = \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix}$; $\quad \Delta = 144 > 0$, $A = f_{xx} < 0 \Rightarrow$ loc. max.

$D^2 f(4,0) = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$; $\quad \Delta = 144 > 0$, $A = f_{xx} > 0 \Rightarrow$ loc. min

$D^2 f(2,2) = \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix}$; $\quad \Delta < 0$ saddle point.

$D^2 f(2,-2) = \begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}$; $\quad \Delta < 0$ saddle point.

Continued...

5. (6 points) Consider the following constrained optimization problem:

$$f(x,y,z) = yz + xy \longrightarrow \text{max/min},$$
$$\text{constraints:} \quad g(x,y,z) = x^2 + y^2 = 1, \quad h(x,y,z) = y^2 + z^2 = 1.$$

(a) Set up a system of equations for the critical points using Lagrange's method. Verify that the number of equations matches the number of unknowns.

$$\vec{\nabla} f = (y, \ x+z, \ y);$$
$$\vec{\nabla} g = (2x, 2y, 0); \quad \vec{\nabla} h = (0, 2y, 2z)$$
$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$$\begin{cases} y = 2\lambda x \\ x+z = 2\lambda y + 2\mu y \\ y = 2\mu z \\ x^2 + y^2 = 1 \\ y^2 + z^2 = 1 \end{cases}$$

5 equations
5 unknowns.

(b) Verify that $(x_0, y_0, z_0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ satisfies the equations of system in part (a). *[You do not need to find all solutions of the system.]*

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}} = 2\lambda \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\lambda \frac{1}{\sqrt{2}} + 2\mu \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} = 2\mu \frac{1}{\sqrt{2}} \end{array}\right\}$$

work when
$\lambda = \frac{1}{2}$
$\mu = \frac{1}{2}$.

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \quad \checkmark$$
$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \quad \checkmark$$

Continued...

6. (8 points) Compute the volume under the graph of the function $z = 10 + x - 2y$, over the rectangle $R = [1, 2] \times [0, 2]$

(a) Exactly, by setting up a double integral and evaluating it using iterated integrals.

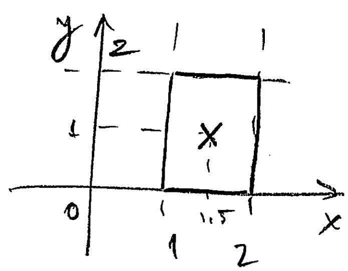
$$V = \int_1^2 \int_0^2 (10 + x - 2y)\, dy\, dx$$

$$= \int_1^2 \left(2 \cdot (10 + x) - 2 \frac{2^2}{2}\right) dx$$

$$= \int_1^2 (16 + 2x)\, dx = 16 + 2\left[\frac{x^2}{2}\right]_1^2$$

$$= 16 + 2\,\frac{2^2 - 1^2}{2} = 16 + 3 = 19.$$

(b) Approximately, by using a Riemann sum with $n = 1$, $m = 1$ and the Midpoint Rule.



Midpoint: $(x_*, y_*) = (1.5, 1)$

$\Delta x = 1$

$\Delta y = 2$

Riemann sum $= \Delta x\, \Delta y\, f(x_*, y_*)$

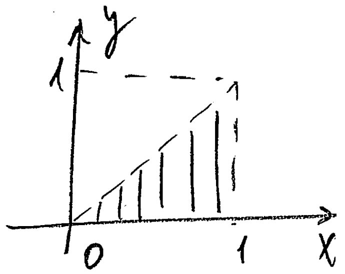
$= 2 \cdot (10 + 1.5 - 2)$

$= 2 \cdot 9.5 = 19.$

7. (6 points) Evaluate the integral by reversing the order of integration:

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx.$$

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx = \iint_{\left\{\begin{smallmatrix} 0 \le x \le 1 \\ x \le y \le 1 \end{smallmatrix}\right\}} e^{x/y}\, dA_{xy}$$



$$= \int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

$$= \int_0^1 \left[y e^{x/y} \right]_{x=0}^{x=y} dy = \int_0^1 (ye - y)\, dy$$

$$= (e-1) \int_0^1 y\, dy = (e-1)\, \frac{1}{2}.$$

8. (bonus: 6 points) If $u = u(x,y)$ and $\tilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$ show that

$$u_{xx} + u_{yy} = \tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \tilde{u}_{\theta\theta}\cdot\frac{1}{r^2}$$

[Hint: the following formulas are handy: $r_x = \frac{x}{r}$, $r_y = \frac{y}{r}$, $\theta_x = -\frac{y}{r^2}$, $\theta_y = \frac{x}{r^2}$.*]*

$$u_x = \frac{\partial}{\partial x}\tilde{u}(r(x,y), \vartheta(x,y)) = \tilde{u}_r r_x + \tilde{u}_\vartheta \vartheta_x = \tilde{u}_r \frac{x}{r} - \tilde{u}_\vartheta \frac{y}{r^2}$$

$$u_y = \frac{\partial}{\partial y}\tilde{u}(r(x,y), \vartheta(x,y)) = \tilde{u}_r r_y + \tilde{u}_\vartheta \vartheta_y = \tilde{u}_r \frac{y}{r} + \tilde{u}_\vartheta \frac{x}{r^2}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\tilde{u}_r \frac{x}{r} - \tilde{u}_\vartheta \frac{y}{r^2}\right)$$

$$= \left(\tilde{u}_{rr} r_x + \tilde{u}_{r\theta}\vartheta_x\right)\frac{x}{r} + \tilde{u}_r \frac{1}{r} - \tilde{u}_r \frac{x}{r^2} r_x$$

$$-\left(\tilde{u}_{\vartheta r} r_x + \tilde{u}_{\vartheta\vartheta}\vartheta_x\right)\frac{y}{r^2} + 2\tilde{u}_\vartheta \frac{y}{r^3} r_x$$

$$= \tilde{u}_{rr}\left(\frac{x}{r}\right)^2 - 2\tilde{u}_{r\theta}\frac{xy}{r^3} + \tilde{u}_{\theta\theta}\left(\frac{y}{r}\right)^2 + u_r\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

$$+ 2u_\vartheta \frac{xy}{r^4}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\tilde{u}_r \frac{y}{r} + \tilde{u}_\vartheta \frac{x}{r^2}\right)$$

$$= \left(\tilde{u}_{rr} r_y + \tilde{u}_{r\theta}\vartheta_y\right)\frac{y}{r} + \tilde{u}_r \frac{1}{r} - \tilde{u}_r \frac{y}{r^2} r_y$$

$$+ \left(\tilde{u}_{\theta r} r_y + \tilde{u}_{\theta\theta}\vartheta_y\right)\frac{x}{r^2} - 2\tilde{u}_\vartheta \frac{x}{r^3} r_y$$

$$= \tilde{u}_{rr}\left(\frac{y}{r}\right)^2 + 2\tilde{u}_{r\theta}\frac{xy}{r^3} + \tilde{u}_{\theta\theta}\left(\frac{x}{r^2}\right)^2$$

$$+ \tilde{u}_r\left(\frac{1}{r} - \frac{y^2}{r^3}\right) - 2u_\vartheta \frac{xy}{r^4}$$

these terms cancel

$$u_{xx} + u_{yy} = \tilde{u}_{rr}\frac{x^2+y^2}{r^2} + \tilde{u}_{\theta\theta}\frac{x^2+y^2}{r^4}$$

$$+ \tilde{u}_r\left(\frac{2}{r} - \frac{x^2+y^2}{r^3}\right) = \tilde{u}_{rr} + \frac{1}{r^2}\tilde{u}_{\theta\theta} + \frac{1}{r}\tilde{u}_r.$$

THE END.