Mon, Oct 5, 2015 **MATH 250, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

This test includes 7 questions with the total of 50 points, on 9 pages. The last page is a formula reference. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) (a) Find an equation for the sphere that passes through the point $A(2, 0, -1)$ and has center at $C(6, -3, -1)$.

$$|AC| = \sqrt{(6-2)^2+(-3-0)^2+(-1-(-1))^2} = \sqrt{4^2+3^2} = 5.$$

Eqn. of the sphere:

$$(x-6)^2+(y+3)^2+(z+1)^2 = 25$$

(b) Does the sphere intersect the yz-plane? If yes, describe the curve of their intersection.

yz-plane: $x = 0$

$$6^2 + (y+3)^2 + (z+1)^2 = 5^2$$

$$(y+3)^2 + (z+1)^2 = -11$$

has no real solution

$\Rightarrow$ the sphere does not intersect the yz-plane.

2. (8 points) (a) Prove that for any two vectors $\vec{a}$, $\vec{b}$ in $\mathbb{R}^3$, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

$$\vec{a} = (a_1, a_2, a_3)\ ; \quad \vec{b} = (b_1, b_2, b_3)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

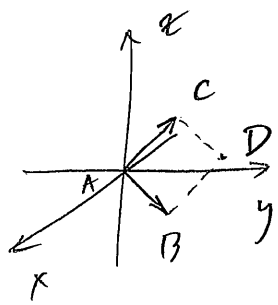
$$-\vec{b} \times \vec{a} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -(b_2 a_3 - b_3 a_2)\vec{i} + (b_1 a_3 - b_3 a_1)\vec{j} - (b_1 a_2 - b_2 a_1)\vec{k}.$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - b_1 a_2)\vec{k}.$$

Geometrically, $\vec{b} \times \vec{a}$ is a vector of same length as $\vec{a} \times \vec{b}$, perpendicular to both $\vec{a}, \vec{b}$, and that forms a left-handed triple $(\vec{a}, \vec{b}, \vec{b} \times \vec{a})$.

(b) Find the area of the parallelogram with the vertices $A(0,0,0)$, $B(1,1,0)$, $C(0,1,1)$, $D(1,2,1)$,

$$\Rightarrow \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}.$$



$$\text{area}(ABCD) = |\vec{AC} \times \vec{AB}|$$

$$= \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \right| = |\vec{i} - \vec{j} + \vec{k}|$$

$$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

3. (8 points) (a) Find an equation of the plane through the points $A(1,1,0)$, $B(1,0,1)$ and $C(0,1,1)$.

$$\vec{AB} = (0,-1,1)\,; \quad \vec{AC} = (-1,0,1)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k} = -(\vec{i}+\vec{j}+\vec{k})$$

Normal vector $\vec{N} = (1,1,1)$

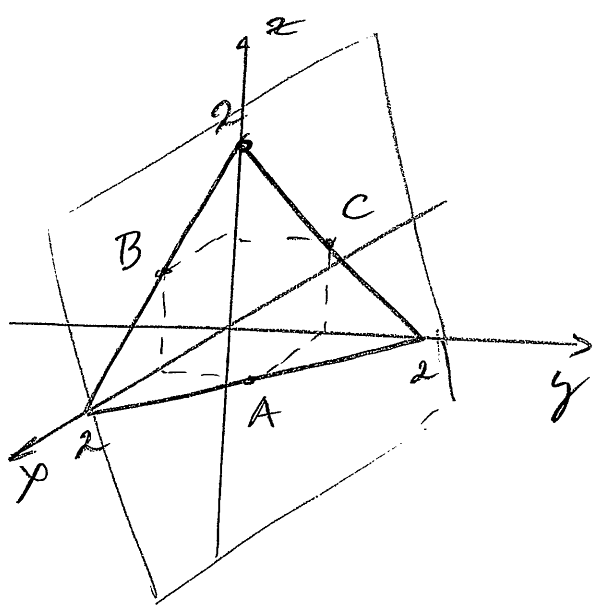
$$(x-1) + (y-1) + z = 0$$

$$x + y + z = 2$$

(b) Find the intercepts of the plane in part (a) on the coordinate axes.

$x = 2$ $(y=z=0)$ $\qquad y = 2$ $(x=z=0)$ $\qquad z = 2$ $(x=y=0)$

(c) Use the intercepts to sketch a graph of the plane in three-dimensional space.



The plane contains the triangle with intercept points as vertices.

4. (6 points) Is the line with symmetric equations

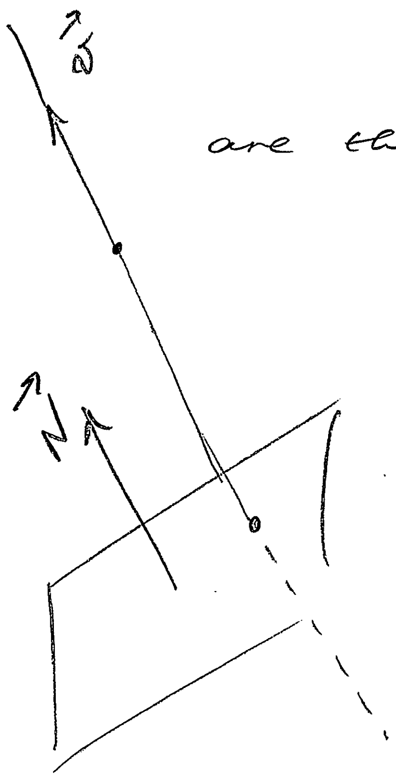
$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-3}{2}$$

parallel to the plane $2x - y + 2z = 2$? Justify your answer. In case they are not parallel, find the point of their intersection.

Direction vector for the line: $\vec{v} = (2, -1, 2)$

Normal vector for the plane: $\vec{N} = (2, -1, 2)$

are the same $\Rightarrow$ the line is perpendicular to the plane.



$\Rightarrow$ They must intersect.

Param. eqns. for the line:

$$\begin{cases} x = 2 + 2t \\ y = -1 - t \\ z = 3 + 2t \end{cases}$$

subst. into $2x-y+2z=2$ $\Rightarrow$ $4 + 4t + 1 + t + 6 + 4t = 2$

$9t + 9 = 0$

$t = -1$

subst. $t=-1$ into param. eqns. $\Rightarrow$ $(x, y, z) = (0, 0, 1)$ – point of intersection.

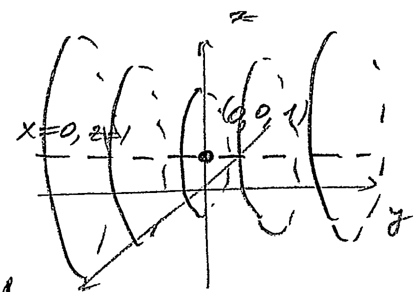
5. (8 points) (a) Reduce the equation to one of the standard forms and classify it:

$$x^2 - 16y^2 + z^2 - 2z = 15.$$

$$x^2 - 16y^2 + (z^2 - 2z + 1) = 16$$

$$x^2 - 16y^2 + (z-1)^2 = 16$$

$$\frac{x^2}{16} - y^2 + \frac{(z-1)^2}{16} = 1$$



$$\rightsquigarrow X^2 - Y^2 + Z^2 = 1 -$$ elliptic hyperboloid of one sheet.

$$X = \frac{x}{4}, \quad Y = y, \quad Z = \frac{z-1}{4}$$

Find the following:

(b) the geometric center of the surface; $(x_0, y_0, z_0) = (0, 0, 1)$

(c) the major axis (the axis of rotational symmetry);

the axis $x=0,\ z=1$ (parallel to the y-axis)

(d) the traces of the surface by the planes perpendicular to the major axis;

$$y = k \implies \frac{x^2}{16} + \frac{(z-1)^2}{16} = k^2 + 1$$

$$x^2 + (z-1)^2 = \left(4\sqrt{k^2+1}\right)^2$$

– circles of radius $4\sqrt{k^2+1}$ about $x=0,\ z=1$

$= 4$ when $k=0$, increasing with $|k|$

(e) the traces by planes containing the major axis and parallel to the coordinate planes.

$$x = 0 \implies -y^2 + \frac{(z-1)^2}{16} = 1$$

hyperbola with asymptotes $z = 1 \pm 4y$

vertices $z = 1 \pm 4 = -3, 5$

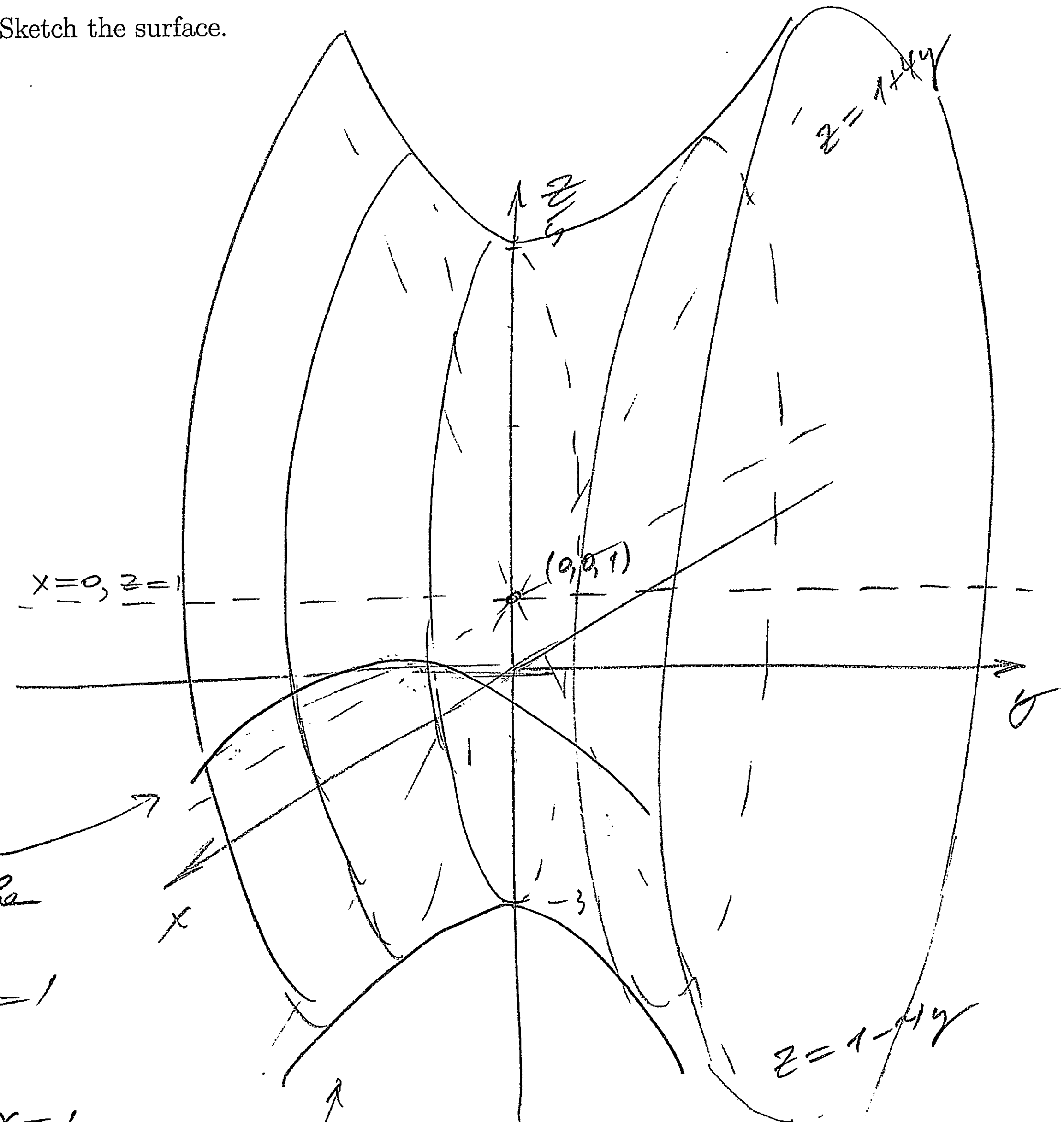
$$z = 1 \implies \frac{x^2}{16} - y^2 = 1$$

Continued...

hyperbola with asymptotes $x = \pm 4y$

vertices $x = \pm 4$

(f) Sketch the surface.



hyperbola
$\frac{x^2}{16} - y^2 = 1$
in the
plane $x=1$
(|| to xy-plane)

hyperbola $-y^2 + \frac{(z-1)^2}{16} = 1$
in the yz-plane

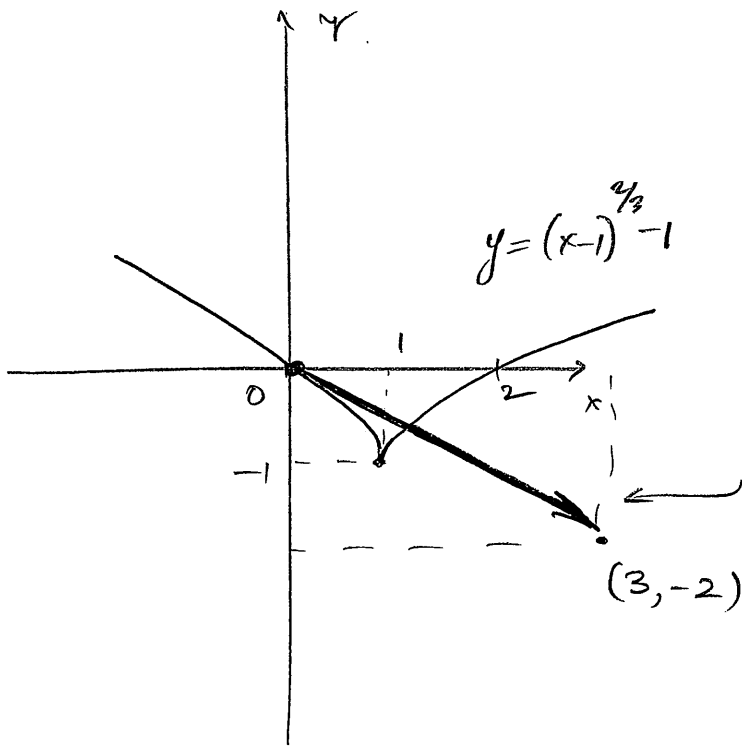
6. (6 points) (a) Sketch the plane curve with the given vector equation:

$$\vec{r}(t) = (t^3 + 1, t^2 - 1).$$

(b) Find the velocity vector $\vec{r}\,'(t)$. Sketch the velocity vector for the value of t when the curve passes through the origin.

$$\begin{cases} x = t^3+1 \\ y = t^2-1 \end{cases} \Rightarrow \quad t^3 = x-1 \Rightarrow t = (x-1)^{1/3}$$

$$\Rightarrow y = (x-1)^{2/3} - 1$$



$r'(t) = (3t^2, 2t)$

Origin: $x=0, y=0$:

$t^3+1=0$

$\Rightarrow t=-1$

(also $(-1)^2-1=0$)

$r'(-1) = (3,-2)$

7. (8 points) Consider the parametric curve:

$$\vec{r}(t) = (t^2, \tfrac{2}{3}t^3, t).$$

$t = 1$!

(a) Find $\vec{r}'(t)$, $|\vec{r}'(t)|$ and the vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ at the point $(1, \frac{2}{3}, 1)$.

$\vec{r}'(t) = (2t, 2t^2, 1)$; $|\vec{r}'(t)| = \sqrt{4t^2+4t^4+1} = \sqrt{(2t^2+1)^2} = 2t^2+1$

$\vec{T}(t) = \frac{1}{2t^2+1}(2t, 2t^2, 1)$, $\vec{T}'(t) = -\frac{4t}{(2t^2+1)^2}(2t, 2t^2, 1) + \frac{1}{2t^2+1}(2, 4t, 0)$

$\boxed{\vec{T}(1) = \frac{1}{3}(2, 2, 1)}$

$\vec{T}'(t) = \frac{1}{(2t^2+1)^2}\left(-8t^2+4t^2+2, -8t^3+8t^3+4t, -4t\right)$

$= \frac{2}{(2t^2+1)^2}(1-2t^2, 2t, -2t)$; $T'(1) = \frac{2}{9}(-1, 2, -2)$

$\vec{N}(1) = \frac{\vec{T}'(1)}{|\vec{T}'(1)|} = \frac{(-1, 2, -2)}{|(-1, 2, -2)|} = \boxed{\frac{1}{3}(-1, 2, -2)}$.

$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \frac{1}{9}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{vmatrix} = \frac{1}{9}(-6, 3, 6) = \boxed{\frac{1}{3}(-2, 1, 2)}$

(b) Find the curvature, and the normal and tangential components of acceleration at the point specified in part (a).

$\vec{r}'(t) = (2t, 2t^2, 1)$; $\vec{r}''(t) = (2, 4t, 0)$

$\vec{r}'(t) \cdot \vec{r}''(t) = 4t + 8t^3 = 4t(1+2t^2)$

$a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|} = \frac{4t(1+2t^2)}{1+2t^2} = 4t$

$\boxed{a_T(t=1) = 4}$

$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t, & 2t^2 & 1 \\ 2 & 4t & 0 \end{vmatrix} = (-4t, 2, 4t^2) = 2(-2t, 1, 2t^2)$

Continued...

$\boxed{a_N(t=1) = \frac{2 \cdot |(-2, 1, 2)|}{3} = 2}$ $\boxed{\kappa(t=1) = \frac{2|(-2,1,2)|}{27} = \frac{2}{9}}$

Reference Page

For a space curve with the position vector $\vec{r}(t)$:

- $\vec{r}'(t)$ – the velocity vector
- $|\vec{r}'(t)|$ – the instantaneous speed
- $s(t) = \int_{t_0}^{t} |\vec{r}'(u)|\, du$ – the arc-length function
- $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ – the unit tangent vector
- $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ – the principal normal
- $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ – the binormal
- $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ – the scalar curvature
- $\vec{a} = \vec{v}' = a_T\,\vec{T} + a_N\,\vec{N}$ – the normal and tangential components of acceleration
- $a_T = \dfrac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$
- $a_N = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$

The end.