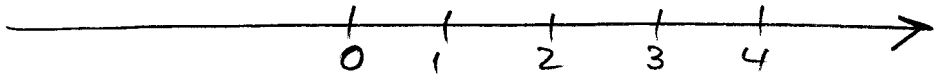
MATH 210 BASIC NUMBER CONCEPTS

Negative numbers and integers.

Whole numbers $= \{0, 1, 2, \dots\}$



Arithmetic operations:

Addition: $a+b$ always works

$a-b$ works only if $a \geq b$

$7-5=2$

$5-7=?$

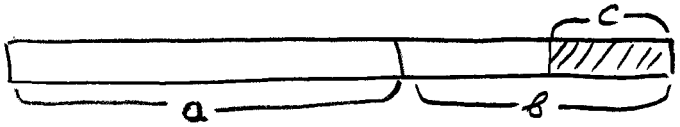
Rules of arithmetic

[here a, b, and c are whole numbers!]

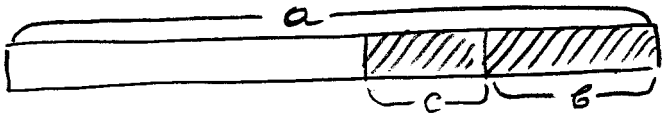
associative

$a+(b+c) = (a+b)+c$

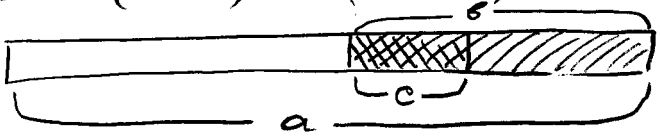
$a+(b-c) = (a+b)-c = "a+b-c"$



$a-(b+c) = (a-b)-c = "a-b-c"$



$a-(b-c) = (a-b)+c = "a-b+c"$



(shaded part is subtracted, cross-shaded is added)

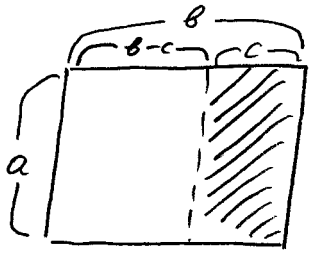
This is valid notation

because the result is independent of the parentheses

distributive

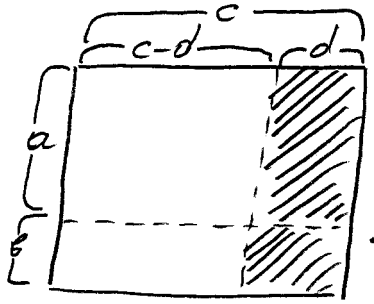
$a \times (b+c) = a \times b + a \times c$

$a \times (b-c) = a \times b - a \times c$

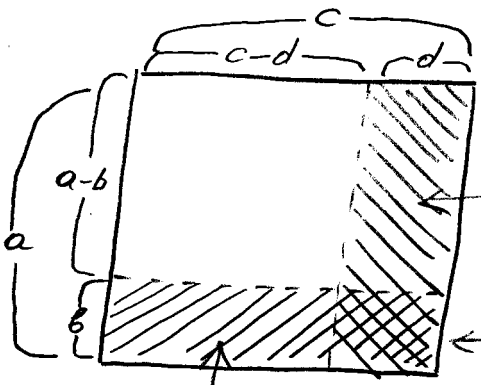


(shaded part is subtracted)

$(a+b) \times (c-d) = \underbrace{ac+bc}_{\text{full rectangle}} - ad - bd$



$(a-b) \times (c-d) = \underbrace{ac}_{\text{full rect.}} - bc - ad + bd$



Shaded part is subtracted; double shaded part is added back.

We wish to extend the system of whole numbers so that <u>subtraction is always possible</u>, and the usual rules of arithmetic are preserved.

Why? — Curiosity: why are we allowed to subtract $7-5$ but not $5-7$?

— "Negative numbers" make sense: if we have \$5 and borrow \$7 then we still owe \$2 (negative \$2)

— Algebra — perhaps, most important reason!

Better wrong or ?

$$60-3x=15$$
$$-3x=15-60$$
$$-3x=-45$$
$$3x=45$$
$$x=15$$

(prefer to keep x on the left)

So, we need to learn to subtract more than we have (credit cards ??)

First we'll learn to subtract from zero:

Define negative numbers as

new numbers!

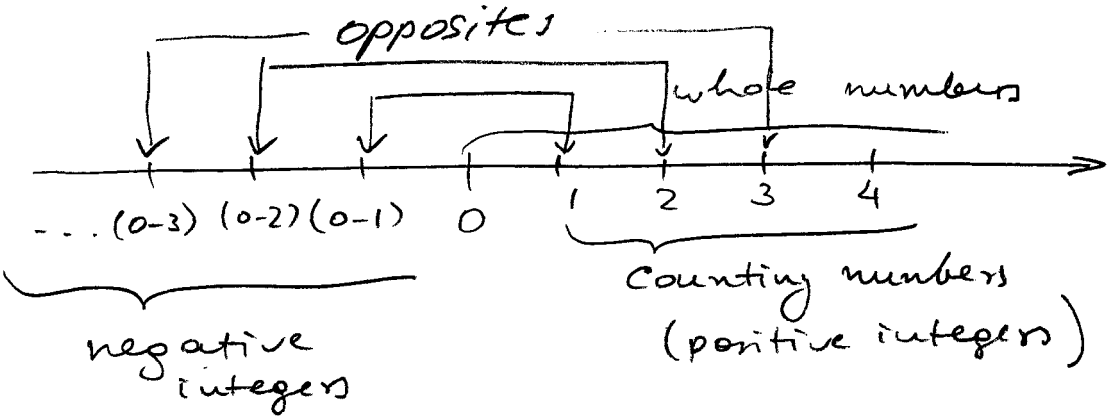
$(0-1)=^{-}1$ or "(-1)" or "-1"

$(0-2)=^{-}2$ or "(-2)" or "-2"

⋮ ⋮

$(0-a)=^{-}a$ or "$(-a)$" or "$-a$"

Each whole number has an opposite $a \leftrightarrow (-a)$; zero is its own opposite.



$$\text{Integers} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$$

(positive integers, negative integers, and zero.)

Working with integers — Integer arithmetic

To <u>define</u> operations with integers we follow distributive rules applied to (0-1), (0-2), etc. and associative

* Addition

$$7 + (-5) = 7 + (0-5) = 7+0-5$$
$$= 7-5 = 2$$

Here we had to regroup and use one of the associative identities. →

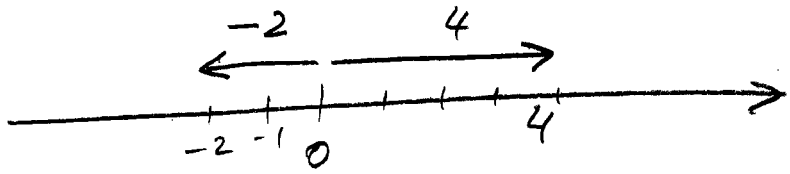
$$5 + (-7) = 5 + (0-7) = 5+0-7$$
$$= 0 - 7 + 5 = 0 - (7-5)$$
$$= 0 - 2 = -2.$$

If the negative amount is more than positive, the sum is negative

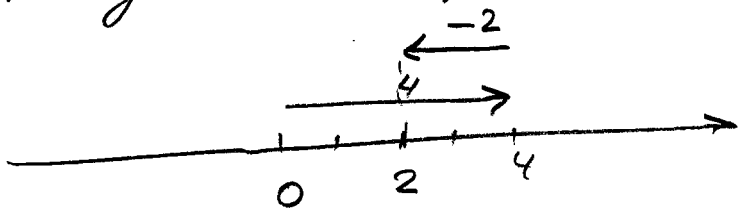
(and equals $-$(larger^{+} − smaller^{+}))

$$(-3) + (-2) = (0-3) + (0-2)$$
$$= 0 - 3 + 0 - 2$$
$$= 0 - (3+2) = 0 - 5 = -5.$$

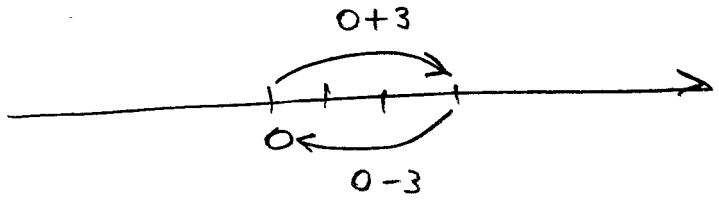
Best illustrated with the "vector model"



Adding means superimposing arrows;



$4+(-2)=2$



$3+(-3)=0$

Rules of the opposites:

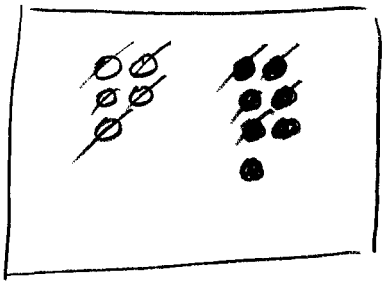
Rule 1: For any integer a, (positive, neg. or zero) $-(-a)=a$

Rule 2: For any int. a, $a+(-a)=0$

$(-a)+a=0$.

Other models:

"(Colored) Clip model" (use with caution!)



white is positive,
black is negative,
positive + negative pairs cancel

$5+(-7)=-2$

Money model: $2 + $2 IOU → zero.

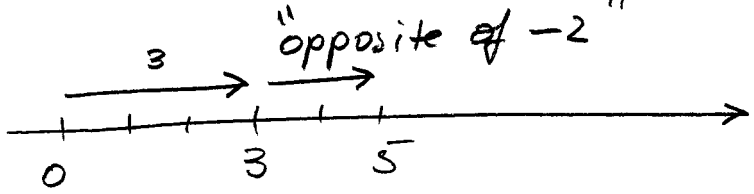
* Subtraction — work with integers according to rules of arithmetic

$$5-7 = 0+5-7 = 0-7+5 =$$
$$= 0-(7-5) = 0-2 = -2.$$

money model: $2 in debt

$$3-(-2) = 3-(0-2) = 3-0+2 = 5$$

Vector model:



"Steps model"

Positive numbers – steps forward
Negative numbers – steps backward
Addition – follow the sequence
Subtraction – turn around, then do prescribed number of steps

— class demonstration —

Rule of subtraction

Rule 3 If a, b are integers (positive, neg. or zero) then

$$a+(-b) = a-b.$$

Subtracting a number is same as adding its opposite.

Multiplication

$3 \times (-4) = 3$ of $(-4) = (-4)$ and (-4) and (-4)

$= (-4) + (-4) + (-4)$

$= -12$

If first factor is positive then the repeated addition interpretation is OK.

(it's OK with zero, too: $0 \times (-4) = 0$)

How about

$(-1) \times 3$?

One way to see it is to require commutative property:

$(-1) \times 3 = 3 \times (-1) = -3.$

Without using commutative:

$-1 \times 3 = (0-1) \times 3 = 0 \times 3 - 1 \times 3 = 0 - 3 = -3$

↑ distributive property.

Another approach:

$0 \times 3 = 0$

$(1 + (-1)) \times 3 = 0$ ↓ distributive property.

$1 \times 3 + (-1) \times 3 = 0$

$3 + (-1) \times 3 = 0$ ↓ subtract 3

$(-1) \times 3 = 0 - 3 = -3$

This bypasses using distributive property for subtraction (only addition is used.)

Exercise: Using distributive properties prove that if a, b — counting numbers

then (i) $a \times (-b)$ is negative

(ii) $(-a) \times b$ is negative

(iii) $(-a) \times (-b)$ is positive

Solution for (ii)

$$(-a) \times b = (0 - a) \times b = \overbrace{0 \times b}^{\text{"zero of } b\text{"}} - a \times b$$

$$= 0 - a \times b = -a \times b$$

negative since a, b - positive (counting numbers)

Rule of multiplication by opposite:

Rule 4 $$(-a) \times b = a \times (-b) = -ab$$

for any integers a, b; positive, negative or zero.

Rule of canceling signs:

$$(-a) \times (-b) = ab$$

Exercise: Give a proof of the canceling signs rule based on Rules 1 and 4.

Solution. (a, b — integers: positive, negative or zero.)

$$(-a) \times (-b) = -(a \times (-b)) = -(-a \times b) = a \times b$$

↑ Rule 4 ↑ Rule 4 ↑ Rule 1

Division

$$\frac{-10}{5} = \frac{0-10}{5} = \frac{0}{5} - \frac{10}{5} = 0-2 = -2$$

(use distributive property on division)

$\frac{-10}{5} = \square$ same as $-10 = 5 \times \square$

$\square = -2.$

(reduce to multiplication)

$\frac{15}{(-3)} = \square$ same as $15 = (-3) \times \square$

$\square = -5$

Visual operations with fractions $\frac{a}{b}$ apply when numerator / denominator are integers ($b \neq 0$)

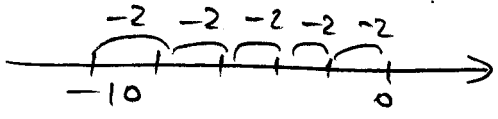
$$\frac{15}{(-3)} = \frac{(-1)\cdot 15}{(-1)\cdot(-3)} = \frac{-15}{3} = -5$$

If dividend is positive, partitive division approach is OK:

-15 is divided into 3 groups; How much on each group? (-5)

If numbers are of same sign, measurement approach is OK:

$\frac{-10}{-2}$: -10 is how many groups of -2? (5)



Negative numbers and exponents.

$$(-3)^2 = (-3)\times(-3) = 9$$

Generally: if n - counting number then

$$a^n = \underbrace{a\times\dots\times a}_{n \text{ times}}$$

$$(-2)^3 = (-2)\times(-2)\times(-2) = -8$$

$$(-2)^5 = -32.$$

$$(-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

Exercise: Prove that if a is an integer (positive, negative or zero)

then $a^2 \geq 0$ ($a^2 = 0$ only if $a = 0$)

(positive or zero)

Solution: If a is positive, it is a counting number, so a^2 is positive.

If a is negative then $a = -b$ where b is positive

then $a^2 = (-b)\times(-b) = b\times b = b^2$ — positive

↑ sign canceling rule

If $a = 0$ then $a^2 = 0$.

Negative exponents are a different matter — see Problems #6,7 (8.1) in the text book

Example: How should we define 2^{-3}?

If rules of exponents were to hold,

then $$2^{-3} \cdot 2^{3} = 2^{-3+3} = 2^{0} = 1$$

so $$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Generally, of a integer, $a \neq 0$, n-counting

define $$a^{-n} = \frac{1}{a^n}.$$

Ex: $$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-6} = \frac{1}{1{,}000{,}000} = 0.000001$$

$$(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$$

$$(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

(negative powers of integers are fractions.)