Wed, Nov 24, 2010 **MATH 150A, Midterm 2** Prof. V. Panferov

Name: (print) ____________________

CSUN ID No. : Solutions.

This test includes 7 questions (57 points in total), on 8 pages. The duration of the test is 1 hour 5 minutes.

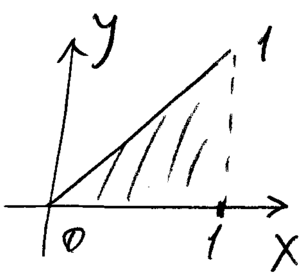
Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

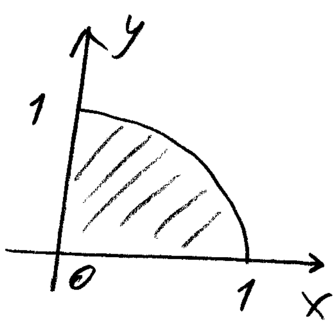
Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Evaluate the integral $\int_0^1 x + \sqrt{1-x^2}\,dx$ by interpreting in terms of areas.

$$\int_0^1 x + \sqrt{1-x^2}\,dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx$$



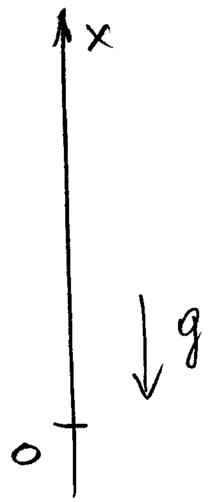
$$\int_0^1 x\,dx = \frac{1}{2}$$



$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

$$\int_0^1 x + \sqrt{1-x^2}\,dx = \frac{1}{2} + \frac{\pi}{4}.$$

2. (8 points) A ball is thrown upward with the initial velocity 40 ft/s. Determine the position of the ball $x(t)$ at time t (assume that the initial position is $x = 0$.) Find the instant of time when the ball will hit the ground. Assume that the acceleration of gravity is $g = 32$ ft/s.^{2}



$x(0) = 0 \quad [ft]$

$x'(0) = 40 \quad [ft/s]$

$x''(t) = -g = -32 \quad [ft/s^2]$

$x'(t) = -32t + 40$

$x(t) = -16t^2 + 40t$

$x(t) = 0 \iff -16t^2 + 40t = 0$

$\iff t = 0$

or $-16t + 40 = 0$

$t = \frac{40}{16} = 2.5 \ [s].$

3. (8 points) For the function $y = \sqrt{x^2-1}$ find the domain, intercepts, asymptotes, critical points, intervals where it increases/decreases, points of local maximum/minimum, inflection points and intervals of concavity upward and downward. Use this information to sketch a graph of the function.

Domain : $x^2-1 \geq 0 \Rightarrow |x| \geq 1 \Rightarrow$
$x \in (-\infty, -1] \cup [1, \infty)$

Intercepts: vertical — none;
horizontal : $x = \pm 1$

Asymptotes: $\sqrt{x^2-1} \approx x + b_1$ when $x > 0$
$\sqrt{x^2-1} \approx -x + b_2$ when $x < 0$

Since $\lim\limits_{x\to\infty} \sqrt{x^2-1} - x = \lim\limits_{x\to\infty} \dfrac{(\sqrt{x^2-1}-x)(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}+x}$

$= \lim\limits_{x\to\infty} \dfrac{-1}{\sqrt{x^2-1}+x} = 0$

and similarly, $\lim\limits_{x\to-\infty} \sqrt{x^2-1} + x = 0$,

the lines $y = x$ and $y = -x$ are oblique asymptotes at $\pm\infty$.

$f(x) = \sqrt{x^2-1} \Rightarrow f'(x) = \dfrac{x}{\sqrt{x^2-1}} \neq 0$ for x in the domain.

$f'(x)$ is undefined at ± 1
$\Rightarrow x = \pm 1$ are crit. points.

$f'(x) > 0$ if $x > 1 \Rightarrow f(x)$ increases on $(1, \infty)$
$f'(x) < 0$ if $x < -1 \Rightarrow f(x)$ decreases on $(-\infty, 1)$

$f''(x) = \dfrac{\sqrt{x^2-1} - \dfrac{x^2}{\sqrt{x^2-1}}}{x^2-1} = \dfrac{-1}{(x^2-1)\sqrt{x^2-1}} < 0$

$\Rightarrow f(x)$ is concave downward on $(-\infty, 1)$ and $(1, \infty)$

There are no inflection pts.

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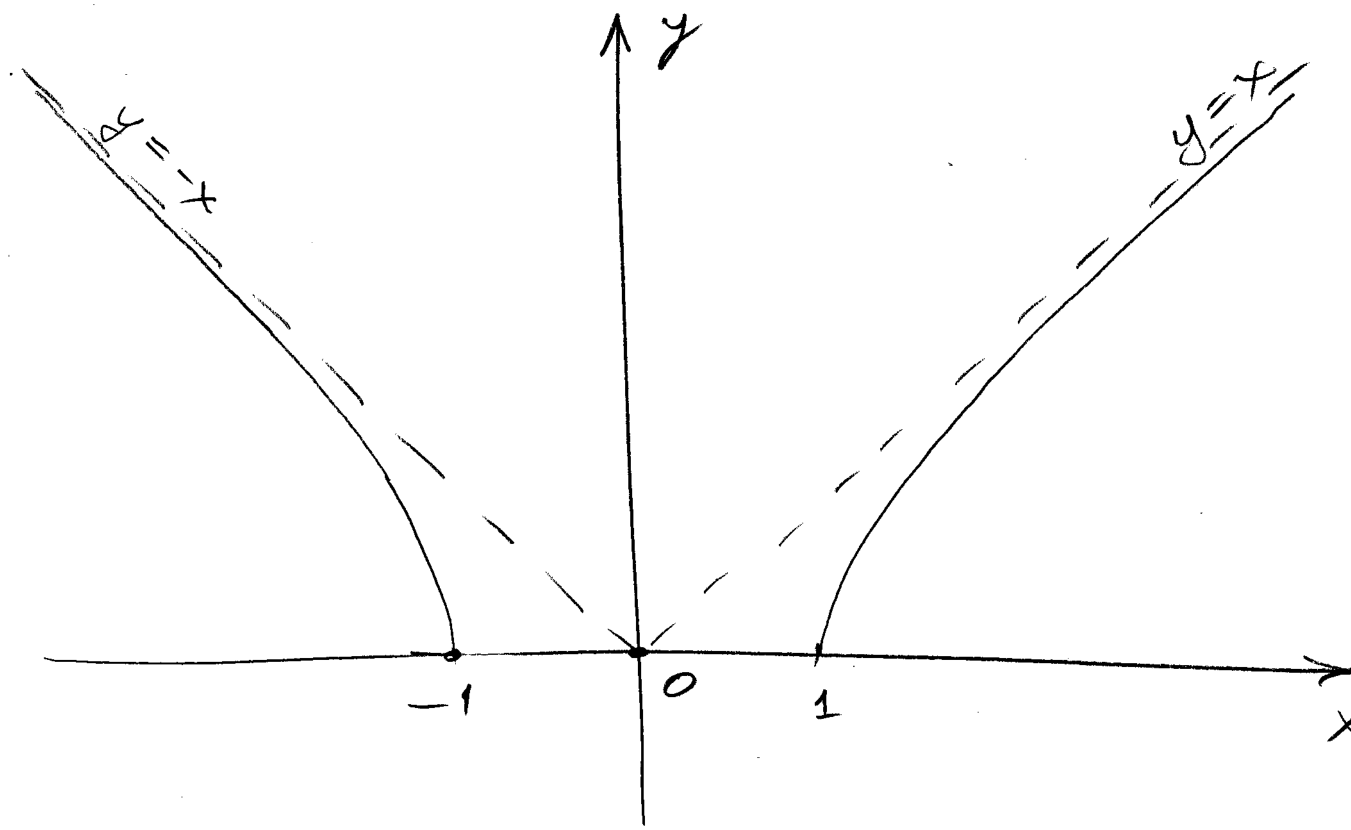


$x = \pm 1$ are points of local and global min.

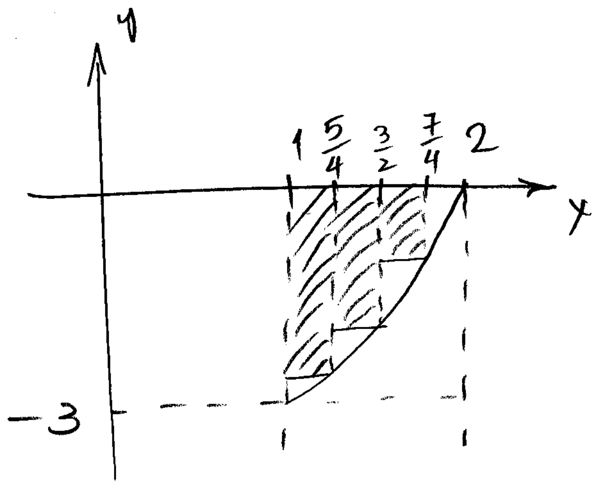
There are no local or global maxima since $f(x)$ is unbounded above.

4. (10 points) (a) Evaluate the Riemann sum for the function $f(x) = x^2 - 4$, $1 \le x \le 2$, with four subintervals, using right end points as sample points. Sketch a graph explaining geometric meaning of the Riemann sum.

$a = 1,\ b = 2$

$n = 4, \quad \Delta x = \frac{b-a}{n} = \frac{1}{4}$

$x_1^* = \frac{5}{4},\ x_2^* = \frac{3}{2},\ x_3^* = \frac{7}{4},\ x_4^* = 2$



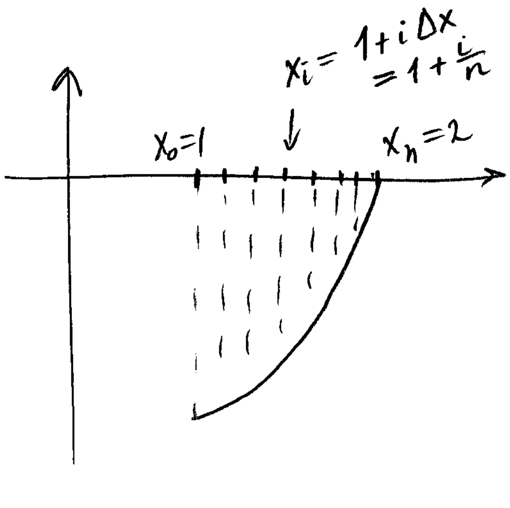
$S_4 = \Delta x \left(f(x_1^*) + f(x_2^*) + f(x_3^*) + \underbrace{f(x_4^*)}_{0}\right)$

$= \frac{1}{4}\left(\left(\frac{5}{4}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{7}{4}\right)^2 - 3 \cdot 4\right)$

$= -1.28125.$

(b) Use the definition of a definite integral using Riemann sums (with right end points) to find the value of the integral $\int_1^2 (x^2 - 4)\,dx$. [Use the identity $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.]

$a = 1$
$b = 2$
$\Delta x = \frac{1}{n}$
$x_i^* = 1 + \frac{i}{n}$



$S_n = \Delta x \sum_{i=1}^{n} f(x_i^*)$

$= \frac{1}{n} \sum_{i=1}^{n} (x_i^*)^2 - 4 = \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 - 4$

$= \frac{1}{n} \sum_{i=1}^{n} 1 + \frac{2i}{n} + \frac{i^2}{n^2} - 4$

$= \frac{1}{n} \sum_{i=1}^{n} -3 + \frac{2}{n} i + \frac{1}{n^2} i^2$

$= -\frac{3}{n} \sum_{i=1}^{n} 1 + \frac{2}{n^2} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} i^2$

$= -3 + \frac{2}{n^2} \frac{n(n+1)}{2} + \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$

$\lim_{n \to \infty} S_n = -3 + \lim_{n \to \infty} \frac{n(n+1)}{n^2} + \frac{1}{6} \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{n^3}$

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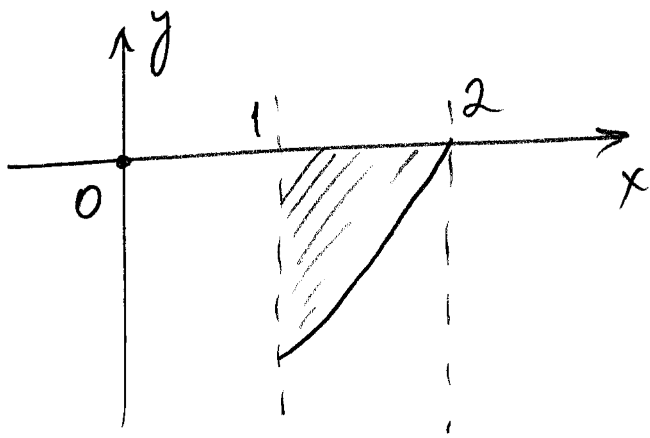
$= -3 + 1 + \frac{1}{6} \cdot 2 = -2 + \frac{1}{3} = -\frac{5}{3}.$

(c) Use the Fundamental Theorem of Calculus to check your answer in part (b).

$$\int_1^2 (x^2-4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{x=1}^{x=2}$$

$$= \frac{8}{3} - 8 - \frac{1}{3} + 4 = -4 + \frac{7}{3} = -\frac{5}{3}$$

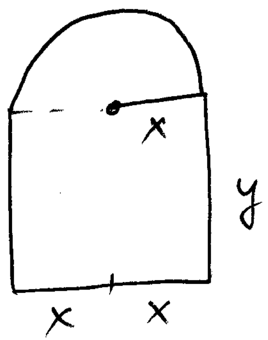
(d) Sketch a graph explaining the geometric meaning of the integral in part (b).



$$\int_1^2 x^2 - 4\,dx$$

$= -$ (Area of the shaded region)

5. (8 points) A window has the shape of a rectangle surmounted by a semicircle (see figure). If the perimeter if the window is 30 ft. find the dimensions of the window so that the greatest possible amount of light is admitted.



perimeter

$$l = 2x + 2y + \pi x$$

Area

$$A = 2xy + \frac{\pi x^2}{2}$$

Problem: Maximize $2xy + \frac{\pi x^2}{2}$

subject to the constraint $(2+\pi)x + 2y = 30$

Using the constraint, $y = \frac{30 - (2+\pi)x}{2}$

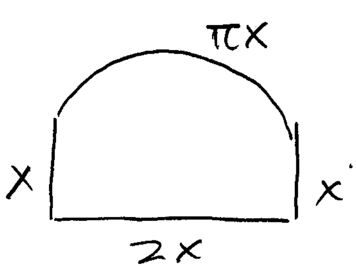
$$A(x) = x\left(30 - (2+\pi)x\right) + \frac{\pi x^2}{2}$$

$$A'(x) = 30 - 2(2+\pi)x + \pi x = 0$$

$$30 - (4+\pi)x = 0$$

$$x = \frac{30}{4+\pi} \approx 4.2 \ [ft]$$

$$y = 15\left(1 - \frac{2+\pi}{4+\pi}\right) = \frac{30}{4+\pi} = x$$



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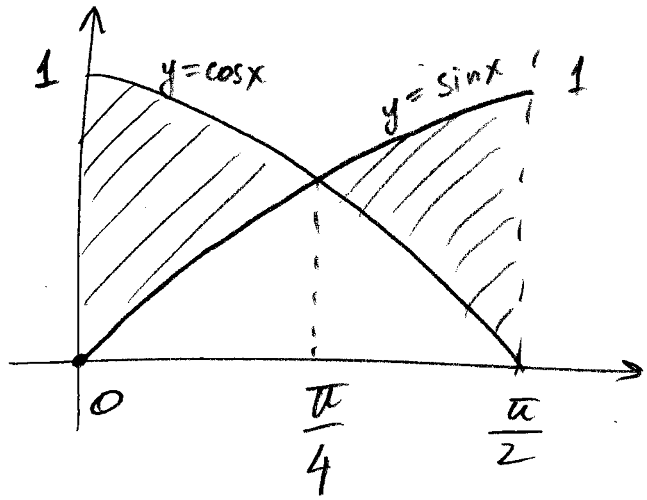
6. (8 points) Find the area of the region between the graphs of $y = \sin x$ and $y = \cos x$, from $x = 0$ to $x = \pi/2$.



$$A = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$\underset{\text{symmetry}}{\overset{\text{by}}{=}} 2 \int_0^{\pi/4} \cos x - \sin x \, dx$$

$$= 2 \left[\sin x + \cos x \right]_{x=0}^{x=\pi/4} =$$

$$= 2 \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \right) = 2(\sqrt{2} - 1).$$

7. (9 points) Find the integrals

(a) $\int y^2 (y - 2\sqrt{y})\, dy$ $= \int y^3 - 2y^{\frac{5}{2}}\, dy$

$= \frac{y^4}{4} - \frac{4}{7} y^{\frac{7}{2}} + C$

By "Fundamental Theorem of Calculus",

(b) $\int_1^5 \frac{dt}{(t-4)^2}$ $= \int_{-3}^{1} \frac{du}{u^2} = \left[-\frac{1}{u} \right]_{u=-3}^{u=1}$

$u = t - 4$

$du = dt$

$t = 1 \Rightarrow u = -3$

$t = 5 \Rightarrow u = 1$

$= -1 - \frac{1}{3} = -\frac{4}{3}$

However, this answer is meaningless, since

$f(t) = \frac{1}{(t-4)^2} > 0.$

Answer: The integral in (b) does not exist; the vertical asymptote at $t = 4$ makes the area undefined ("$+\infty$").

(c) $\int_{1/2}^{1} \frac{\cos(x^{-2})}{x^3}$

$u = x^{-2}$

$du = -2x^{-3} dx$

$x^{-3} dx = -\frac{1}{2} du$

$x = \frac{1}{2} \Rightarrow u = 4$

$x = 1 \Rightarrow u = 1$

$\int_{1/2}^{1} \frac{\cos(x^{-2})}{x^3} dx$

$= -\frac{1}{2} \int_4^1 \cos u \, du$

$= \frac{1}{2} \int_1^4 \cos u \, du$

$= \frac{1}{2} \left[\sin u \right]_{u=1}^{u=4}$

$= \frac{1}{2} (\sin 4 - \sin 1) \approx -0.7991.$

The end.