6.3 $\underline{\qquad\qquad}\ E\ (n_1)$ The total energy is U

$U = n_0(0) + n_1 E = n_1 E$

$\underline{\qquad\qquad}\ 0\ (n_0)$ and $n_0 + n_1 = N$

a) $\Omega(E)$ is the number of all the possible microscopic states of the system when its energy is $U = n_1 E$. As the particles are distinguishable, the number of ways one may choose n_1 objects from a total of N is:

$$\Omega(E) = N! / (N-n_1)!\, n_1!$$

where $n_1 = U/E$. The entropy is:

$$S(U) = k_B \ln \Omega(U) = k_B \ln \frac{N!}{(N-n_1)!\, n_1!} = S(n_1)$$

$\{$Note that $S(n_1=0) = S(n_1=N) = 0$. $\}$

$$\Rightarrow S(U) = k_B\{\ln N! - \ln(N-n_1)! - \ln n_1!\}$$

If one use Stirlings approximation for large n

$$\ln n! = n \ln n - n$$

$$\Rightarrow S(U) = k_B\{N\ln N - N - (N-n_1)\ln(N-n_1) + (N-n_1) - n_1 \ln n_1 + n_1\}$$

$$= k_B\{N\ln N - (N-n_1)\ln(N-n_1) - n_1 \ln n_1\} =$$

$$= k_B\left\{N\ln N - \left(N-\frac{U}{E}\right)\ln\left(N-\frac{U}{E}\right) - \frac{U}{E}\ln\frac{U}{E}\right\}$$

$$\Rightarrow \quad S(U) = k_B \left\{ N \ln \frac{N}{N-n_1} + n_1 \ln \frac{N-n_1}{n_1} \right\} =$$

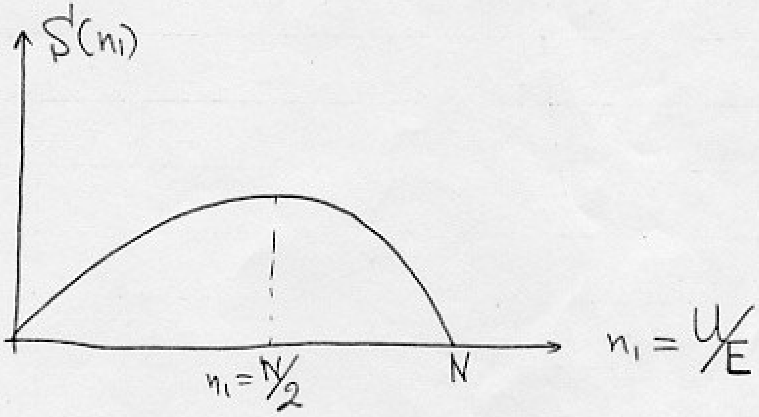
$$= k_B \left\{ N \ln \frac{N}{N-n_1} - n_1 \ln \frac{n_1}{N-n_1} \right\}$$

where $n_1 = U/E$

(b) Note that $\left(\frac{dS}{dn_1}\right) = 0$ gives $\frac{N}{N-n_1} - 1 - \ln n_1 - \frac{n_1}{N-n_1} + \ln(N-n_1) = 0$

Therefore, $S = S_{max}$ when $n_1 = N/2$

A plot of $S(U) = S(n_1)$



(b) Since $U = n_1 E$ is fixed, (microcanonical ensemble), that fixes n_1 and n_0.
The most probable value of n_1 is n_1
" " " " of n_0 is n_0.
The mean square fluctuations of these quantities is zero.

c) $$\beta = \frac{1}{k_B T} = \left(\frac{\partial \ln(\Omega(U))}{\partial U}\right)_N = E^{-1}\left(\frac{\partial \ln \Omega(n_1)}{\partial n_1}\right)_N$$

From the expression above for $\ln \Omega(n_1)$ we have :

$$\beta = \frac{1}{k_B T} = E^{-1}\left\{ \ln(N-n_1) + 1 - 1 - \ln n_1 \right\} = E^{-1} \ln \frac{N-n_1}{n_1} =$$

$$= E^{-1} \ln\left(\frac{N}{n_1} - 1\right)$$

Thus: $\beta E = \ln\left(\frac{N}{n_1} - 1\right)$ or $k_B T = \dfrac{E}{\ln\left(\frac{N}{n_1} - 1\right)} = \dfrac{E}{\ln\left(\frac{NE}{U} - 1\right)}$

Note that when $n_1 = \frac{N}{2}$ $(S = S_{max})$ $\beta \to 0$ and $T \to \infty$

Note that when $\frac{N}{n_1} - 1 < 1$ or $n_1 > \frac{N}{2}$
the temperature is negative.
This corresponds to the portion of the graph with negative slope.

d) The rease that $T<0$ here is that the energy level of a single particle has an upper limit (E). For a gas system, the energy level of a particle does not have an upper limit, and the entropy is an increasing function of U; hence negative temperature cannot occur.
From the point of view of energy, we can say that a system with $T<0$ is "hotter" than any system with $T>0$.
As one starts at the origin in the previous figure and go to the right, we proceed in the direction of increasing U, increasing hotness, and therefor increasing T. At the position of maximum entropy, where both energy levels are

equally populated, the temperature is infinite. Beyond the maximum the temperature must be hotter than ∞. Hence, $T<0$ are hotter than infinity! Thus heat can be extracted from a negative-temperature reservoir with no other effect than the performance of an equivalent amount of work.