# HOMEWORK



12.5 For the ideal Bose gas at high $T > T_c$ $(z \ll 1)$ we have:

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} g_{5/2}(z)$$

and $\rho\lambda^3 = g_{3/2}(z)$ where $g_n(z) = \sum_{\ell=1}^{\infty} z^\ell / \ell^n$.

If we express $z = \sum_{n=0}^{\infty} C_n (\rho\lambda^3)^n$ (as we did in class for the ideal fermion system) we find that:

$$C_0 = 0 \quad , \quad C_1 = 1 \quad \text{and} \quad C_2 = -\frac{1}{2^{3/2}} \quad \text{and} \quad C_3 = -\frac{1}{3^{3/2}}$$

Thus: $$z = \rho\lambda^3 - \frac{1}{2^{3/2}}(\rho\lambda^3)^2 - \frac{1}{3^{3/2}}(\rho\lambda^3)^3 + \cdots$$

Thus:

$$\frac{P}{\rho k_B T} = \frac{1}{\rho\lambda^3}\left[ z + \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \cdots \right] =$$

$$= \frac{1}{\rho\lambda^3}\Bigg\{ \left[ \rho\lambda^3 - \frac{1}{2^{3/2}}(\rho\lambda^3)^2 - \frac{1}{3^{3/2}}(\rho\lambda^3)^3 + \cdots \right]$$

$$+ \frac{1}{2^{5/2}}\left[ \rho\lambda^3 - \frac{1}{2^{3/2}}(\rho\lambda^3)^2 - \frac{1}{3^{3/2}}(\rho\lambda^3)^3 + \cdots \right]^2$$

$$+ \frac{1}{3^{5/2}}\left[ \rho\lambda^3 - \frac{1}{2^{3/2}}(\rho\lambda^3)^2 - \cdots \right]^3 + \cdots \Bigg\}$$

Collecting terms to order $(\rho\lambda^3)^3$ we have:

$$\frac{P}{\rho k_B T} = \frac{1}{\rho\lambda^3}\left\{ \rho\lambda^3 - \frac{1}{2\sqrt{2}}(\rho\lambda^3)^2 - \frac{1}{3\sqrt{3}}(\rho\lambda^3)^3 + \frac{1}{4\sqrt{2}}\left((\rho\lambda^3)^2 - \frac{1}{\sqrt{2}}(\rho\lambda^3)^3 + \dots\right) + \frac{1}{9\sqrt{3}}\left((\rho\lambda^3)^3 + \dots\right)\right\}$$

$$\Rightarrow \quad \frac{Pv}{k_B T} = \frac{1}{\rho\lambda^3}\left\{ \rho\lambda^3 + \left(\frac{1}{4\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)(\rho\lambda^3)^2 + \left(-\frac{1}{3\sqrt{3}} - \frac{1}{8} + \frac{1}{9\sqrt{3}}\right)(\rho\lambda^3)^3 + \dots\right\}$$

Thus:

$$\boxed{\frac{Pv}{k_B T} = 1 - \frac{1}{4\sqrt{2}}(\rho\lambda^3) - \left(\frac{1}{8} + \frac{2}{9\sqrt{3}}\right)(\rho\lambda^3)^2 + \dots}$$

12.6 For a two dimensional Bose ideal gas calculate the $\langle N \rangle / A$ as function of $z$ and $T$.

The density of states for an ideal Bose gas in two dimension is : $N(\epsilon) = \frac{m}{\hbar^2} \frac{L^2}{(2\pi)^2} = \frac{2\pi m A}{h^2}$,

that is independent of energy.

$$\langle N \rangle = \sum_{\vec{p}} \langle n_{\vec{p}} \rangle = \sum_{\vec{p}} \frac{z\, e^{-\beta \epsilon_{\vec{p}}}}{1 - z\, e^{-\beta \epsilon_{\vec{p}}}} = \sum_{\vec{p}} \frac{1}{z^{-1} e^{\beta \epsilon_{\vec{p}}} - 1}$$

Converting that in an integral over energy we have

$$\langle N \rangle = \int_0^\infty N(\epsilon) \langle n(\epsilon) \rangle d\epsilon = \frac{2\pi m A}{h^2} \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} =$$

$$= \frac{2\pi m A}{h^2} \int_0^\infty \left( \sum_{\ell=1}^\infty e^{-\ell \beta (\epsilon - \mu)} \right) d\epsilon =$$

$$= \frac{2\pi m A}{h^2} k_B T \sum_{\ell=1}^\infty \frac{1}{\ell} e^{\ell \beta \mu}$$

Since the Bose - Einstein condensation occurs when $\mu = 0$, if $\mu = 0$ the above expression diverges. Hence $\mu \neq 0$ and Bose - Einstein condensation does not occur.

# HOMEWORK

1) Calculate the internal energy per particle, $U/N$, the entropy per particle, $S/N$, and the specific heat for an ideal Bose gas for $T > T_c$ and $T < T_c$

In class we derived that

$$\frac{p}{k_B T} = \begin{cases} \frac{1}{\lambda^3} g_{5/2}(z) & \text{for } T > T_c \\ \frac{g_{5/2}(z=1)}{\lambda^3} & \text{for } T < T_c \end{cases}$$

(a) Since $U = -\left(\frac{\partial \ln \Xi}{\partial \beta}\right)_{\beta\mu} = -\frac{\partial}{\partial \beta}\left[-\ln(1-z) + \frac{V}{\lambda^3} g_{5/2}(z)\right]_{\beta\mu}$

$$= -\frac{\partial}{\partial \beta}\left\{\frac{V}{\lambda^3} g_{5/2}(z)\right\}_{\beta\mu} = -V g_{5/2}(z) \frac{\partial}{\partial \beta}\left(\lambda^{-3}\right) =$$

$$= \frac{3}{2} \frac{k_B T V}{\lambda^3} g_{5/2}(z)$$

Thus, for $T > T_c \Longrightarrow$ $\boxed{\frac{U}{N} = \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(z)}$

For $T < T_c$, $z = 1 \Longrightarrow$

$$\boxed{\frac{U}{N} = \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(1)}$$

(b) The entropy can be calculated from
$$U = TS - PV + \mu \langle N \rangle$$
Since $PV = 2U/3$, it follows that

$$S = \frac{5U}{3T} - \frac{\mu \langle N \rangle}{T} = \frac{5U}{3T} - \langle N \rangle k_B \ln z$$

Since we know $U$, we find that: $(\langle N \rangle = N)$

$$\frac{S}{k_B N} = \frac{5}{2} \frac{U}{\lambda^3} g_{5/2}(z) - \ln z \quad \text{for } T > T_c .$$

For $T < T_c$, $z = 1 \Rightarrow$

$$\frac{S}{k_B N} = \frac{5}{2} \frac{U}{\lambda^3} g_{5/2}(1)$$

(c) In order to calculate $C_V$, it is necessary first to examine the derivative of $z$ with respect to $T$, which is obtained from $\frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z)$

$\Rightarrow$ Differentiation with respect to $T$ at constant volume gives:
$$\left(\frac{\partial z}{\partial T}\right)_V = -\frac{3z}{2T} \frac{g_{3/2}(z)}{g_{1/2}(z)}$$
From that and $C_V = T\left(\frac{\partial S}{\partial T}\right)_V$ we find

$$\frac{C}{Nk_B} = \begin{cases} \dfrac{15}{4}\dfrac{v}{\lambda^3}\, g_{5/2}(z) - \dfrac{9}{4}\dfrac{g_{3/2}(z)}{g_{1/2}(z)} & T > T_c \\[2ex] \dfrac{15}{4}\dfrac{v}{\lambda^3}\, g_{5/2}(1) & T < T_c \end{cases}$$

3. Since $dA = -TdS - pdV \implies p = -\left(\frac{\partial A}{\partial V}\right)_T$

When $T = 0\,K$, $A = U$ and $p = -\left(\frac{\partial U}{\partial V}\right)_T$.

Using $pV = \frac{2}{3}U$, we have:

$$p = -\left(\frac{\partial U}{\partial V}\right)_T = -\left[\frac{\partial}{\partial V}\left(\frac{3}{2}pV\right)\right] = -\frac{3}{2}\left[V\left(\frac{\partial p}{\partial V}\right)_T + p\right]$$

or $$V\left(\frac{\partial p}{\partial V}\right)_T = -\frac{5}{3}p$$

Thus: $$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{3}{5p} \quad \text{at } T = 0\,K.$$

Thus, one has to calculate $p$ in terms of the quantities given, namely the number density and the Fermi energy.

$$p = \frac{2U}{3V} = \frac{2}{3V}\, 2\, \frac{V}{(2\pi)^3} \int_{k<k_F} d^3k\, \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k_F^5}{15 m \pi^2}$$

Similarly $$N = nV = 2\frac{V}{(2\pi)^3}\int_{k<k_F} d^3k$$

$$\implies n = k_F^3/3\pi^2$$

Since $\epsilon_F = \frac{\hbar^2 k_F^2}{2m} \implies p = \frac{2}{5} n \epsilon_F$ at $T = 0\,K$

$$\implies \boxed{\kappa = \frac{3}{2n\epsilon_F}} \quad \text{at } T = 0\,K.$$