Finite Elements in Two Dimensions

Larry Caretto
Mechanical Engineering 501B

Seminar in Engineering Analysis

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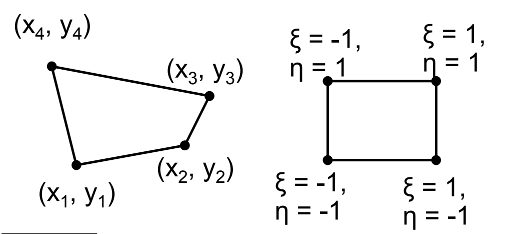
Outline

- Review last week's lectures on introduction to finite elements
- Two dimensional elements and shape functions
  - Triangular and quadrilateral elements with linear and higher orders
- Derivation of algebraic equations
- Examples of results

Review Finite Element Basics

- Designed for 2D and 3D geometries
- Use for 1D case as example
- Basic idea is to divide region into small elements (line, area, volume)
- Use interpolating polynomial for each element
  - Represent both geometry (independent variables) and dependent variable
  - Polynomials called basis functions or shape functions like ϕ_i in previous slide

Review Shape (Basis) Functions



Note: $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Review Shape Function Use

- Model geometry (x, y) and dependent variable, u
- Shape functions associated with element nodes such that $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$x = \sum_{i=1}^{4} x_i\varphi_i \qquad y = \sum_{i=1}^{4} y_i\varphi_i \qquad \hat{u} = \sum_{i=1}^{4} u_i\varphi_i$$

- u_i are values of dependent variable at nodes in grid that we want to find

Review Galerkin

- Method of weighted residuals

$$\int_\Omega w_i\left[L(\hat{u}) - b\right]d\Omega = 0 \quad i = 0,\ldots N$$

- Galerkin method uses shape function for weighting function $w_i = \phi_i$

$$\int_\Omega \phi_i\left[L(\hat{u}) - b\right]d\Omega = 0 \quad i = 0,\ldots N$$

- Have to get equations for u_i $\quad \hat{u} = \sum_{i=0}^{N} u_i\phi_i(\mathbf{x})$

Review Galerkin Example

- One-dimensional example L(u) = $d^2u/dx^2 + a^2u = 0 = b$ for $0 \le x \le L$

$$\int_0^L \phi_i(x)\left[\frac{d^2\hat{u}}{dx^2} + a^2\hat{u}\right]dx = 0 \qquad i = 0,\dots N$$

- Get following result

$$\left[\varphi_i \frac{d\hat{u}}{dx}\right]_0^L = \sum_{j=0}^{N} A_{ij}u_j \qquad i = 0,\dots N$$

$$A_{ij} = \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} + a^2\varphi_i\varphi_j\right]dx$$

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Review Linear Shape Functions

$$\varphi_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

Get needed shape functions by substituting i-1 and i +1 for i

Substitute shape functions and derivatives into integrals

$$\frac{d\varphi_i(x)}{dx} = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{1}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{-1}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

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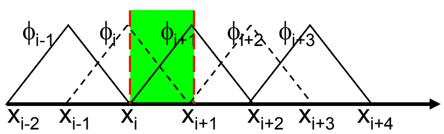
Review Integration

- Two approaches to 1D example
  - Look at integrals over region
  - Element analysis (with same integrals)



- For 2D and 3D element analysis is easier approach

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Review Assembly

- Element approach gives equations for typical element with local node numbers
- Assembly of element results accounts for nodes on more than one element
- In 1D case element has two nodes
  - Except for boundary elements, each element node is present on two different elements
  - Add equations for individual node from both elements

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Review Linear Results

$$\alpha_i = \frac{a^2}{3}(x_{i+1} - x_i) - \frac{1}{x_{i+1} - x_i} \qquad \beta_i = \frac{a^2}{6}(x_{i+1} - x_i) + \frac{1}{x_{i+1} - x_i}$$

$$A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} = 0$$

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1}\right]dx = \beta_{i-1}$$

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i\right]dx = \alpha_i + \alpha_{i-1}$$

$$A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i+1}\right]dx = \beta_i$$

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Review Boundary Gradients

- If we have Dirichlet boundary conditions, we can solve for u values then find gradients
- For Neumann or mixed boundary conditions, we must include gradients in tridiagonal solution
- Write boundary conditions as a du/dx + b u = c and make $g_0 = du/dx|_{x=0}$ the first variable and $g_L = du/dx|_{x=L}$ the last one

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Review Assembled Equations

- Equations below only handle boundary conditions with gradients ($a_0 \neq 0$)

$$\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \alpha_0 & \beta_0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \beta_0 & \alpha_0+\alpha_1 & \beta_1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \beta_1 & \alpha_1+\alpha_2 & \beta_2 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \ddots & \beta_{N-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \beta_{N-1} & \alpha_{N-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & b_N & a_N \end{bmatrix} \begin{bmatrix} g_0 \\ u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \\ u_N \\ g_N \end{bmatrix} = \begin{bmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ c_N \end{bmatrix}$$

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Review Errors for a = 2

N	100	100	10	10
Method	FD	FE	FD	FE
e_{RMS}	1.7×10^{-5}	1.7×10^{-5}	1.8×10^{-3}	1.8×10^{-3}
e_{max}	2.4×10^{-5}	2.4×10^{-5}	2.4×10^{-3}	2.4×10^{-3}
$e_{grad}(0)$	3.6×10^{-4}	7.0×10^{-5}	3.6×10^{-2}	7.0×10^{-3}
$e_{grad}(L)$	2.1×10^{-4}	9.6×10^{-5}	1.8×10^{-2}	9.5×10^{-3}

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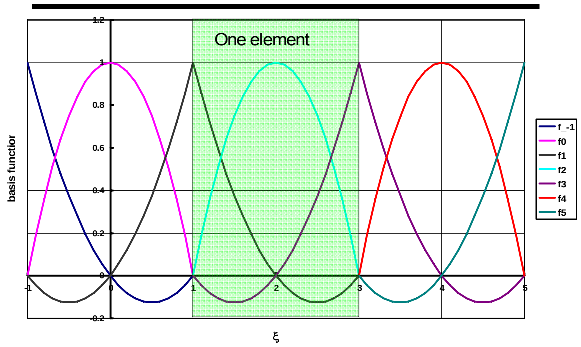
Quadratic Basis Functions, Global Basis



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Review Quadratic Results

- Solve $d^2u/dx^2 + a^2u = 0$ using quadratic shape functions
- 4^{th} order error by log error vs. log h plot

Number of Elements	Element width, h	Maximum Error	RMS Error
5	0.2	3.70E-05	2.40E-05
10	0.1	2.40E-06	1.50E-06
20	0.05	1.50E-07	9.10E-08
40	0.025	9.50E-09	5.90E-09

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2D Finite Elements

- Use same process as in 1D $\quad \hat{u} = \sum_{i=0}^{N} u_i \phi_i(\mathbf{x})$
  – Start with Galerkin formulation

$$\int_\Omega \phi_i \left[L(\hat{u}) - b\right] d\Omega = 0 \quad i = 0, \ldots N$$

  – Use dΩ = dxdy and substitute actual partial differential equation for L(u) = b
  – Use Green's theorem, a two-dimensional analog of integration by parts
  – Substitute $\hat{u}$ and integrate results for appropriate elements and basis functions

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Result of Derivation

- Set of simultaneous linear equations to be solved for all nodes in 2D region

$$\sum_{i=0}^{N} A_{ki} u_i = \begin{pmatrix} boundary \\ terms\ or\ 0 \end{pmatrix} \quad k = 0, \ldots N$$

- Look at example problem that is 2D extension of 1D problem from last week

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + a^2 u = 0$$

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Preview of Result for Example

- Coefficient in linear equations

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \xi}\left[\frac{x_\eta^2 + y_\eta^2}{J}\frac{\partial \varphi_k}{\partial \xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_i}{\partial \eta}\right] d\xi d\eta +$$

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \eta}\left[\frac{x_\xi^2 + y_\xi^2}{J}\frac{\partial \varphi_k}{\partial \eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_k}{\partial \xi}\right] d\xi d\eta$$

$$-\int_{-1}^{1}\int_{-1}^{1} \varphi_k a^2 \phi_i J d\xi d\eta$$

ξ and η are coordinates
ϕ_m are shape functions
J is Jacobian determinant
x_η, y_η, x_ξ, and y_ξ are coordinate derivatives

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2D Example

- Use Helmholtz equation, a 2D analog of the 1D example used last week

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + a^2 u = 0$$

- Galerkin's method then gives

$$\int_\Omega \varphi_k(x)\left[\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + a^2 \hat{u}\right] dxdy = 0 \qquad k = 0, \ldots N$$

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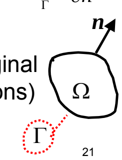
Green's Theorem

- Given here with no derivation
  - See notes on vector calculus for details
  - Region, Ω, has bounding curve, Γ, and $\partial u/\partial n$ is outward gradient normal to Γ

$$\int_\Omega \varphi\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\,dy = -\int_\Omega \left(\frac{\partial \varphi}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial u}{\partial y}\right) dxdy + \oint_\Gamma \varphi \frac{\partial u}{\partial n} ds$$

- Apply this to example problem
  - Allows linear shape functions (original formula gives zero for such functions)
  - Introduces gradient boundaries



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Result and Next Steps

- Application of Green's theorem gives

$$\int_\Omega \varphi_k\left[\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + a^2 \hat{u}\right] dxdy = -\int_\Omega \left[\frac{\partial \hat{u}}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \hat{u}}{\partial y}\frac{\partial \varphi_k}{\partial y}\right] dxdy$$

$$+\int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds + \int_\Omega \varphi_k a^2 \hat{u} dxdy = 0 \qquad k = 0, \ldots N$$

- Substitute equation for $\hat{u} = \sum_{i=0}^{N} u_i \phi_i(\mathbf{x})$
- Select element (quadrilateral or triangle)
- Select shape functions
- Use local coordinate transform

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Substitute $\hat{u} = \sum_{i=0}^{N} u_i \phi_i(\mathbf{x})$

- Multiply previous result by –1 and substitute approximate relationship for u

$$\int_\Omega \left[\frac{\partial \hat{u}}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \hat{u}}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \hat{u}\right] dxdy - \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds = 0 \qquad k = 0, \ldots N$$

$$\int_\Omega \left[\frac{\partial \sum_{i=0}^{N} u_i \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \sum_{i=0}^{N} u_i \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \sum_{i=0}^{N} u_i \phi_i\right] dxdy - \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds = 0$$

$$\sum_{i=0}^{N} u_i \left\{\int_\Omega \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \phi_i\right] dxdy\right\} - \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds = 0$$

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Selecting Elements

- Have general two-dimensional result

$$\sum_{i=0}^{N} u_i \left\{\int_\Omega \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \phi_i\right] dxdy\right\} = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds$$

$$k = 0, \ldots, N$$

- Apply this equation to elements
- Select quadrilateral or triangle
- Different x and y values in coordinate transformations for each element

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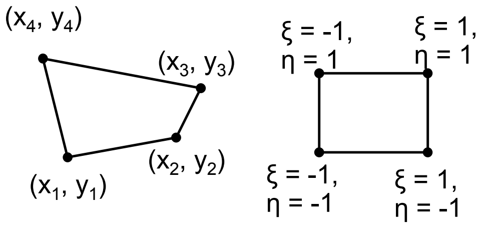
Quadrilateral Element



- Transform integral for quadrilateral in x and y to integral for square in ξ and η

Quadrilateral Transformations

- Transform derivatives and integral in Galerkin result from x-y to ξ-η
  - Other transforms used for triangles
  - For any variable, g(x,y), we use

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \frac{\partial g}{\partial y} = \frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \xi}\xi_x + \frac{\partial g}{\partial \eta}\eta_x \qquad \frac{\partial g}{\partial y} = \frac{\partial g}{\partial \xi}\xi_y + \frac{\partial g}{\partial \eta}\eta_y$$

Notation: $\partial z/\partial w = z_w$

Quadrilateral Transformations II

- Apply transforms to $g = \phi_m$ in integrals

$$\frac{\partial \phi_m}{\partial x} = \frac{\partial \phi_m}{\partial \xi}\xi_x + \frac{\partial \phi_m}{\partial \eta}\eta_x \qquad \frac{\partial \phi_m}{\partial y} = \frac{\partial \phi_m}{\partial \xi}\xi_y + \frac{\partial \phi_m}{\partial \eta}\eta_y$$

$$\sum_{i=0}^{N} u_i \left\{ \int_\Omega \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \phi_i \right] dxdy \right\} = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds$$

- Define integrals as A_{ki} and evaluate for different shape functions

$$\sum_{i=0}^{N} A_{ki} u_i = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds \quad k = 0, \ldots, N$$

Quadrilateral Transformations III

- Have to get derivatives ξ_x, ξ_y, η_x, and η_y from basic relation between x,y and ξ, η
- Geometrical coordinates are given in terms of shape functions
- Shape functions are functions of ξ and η
- Start with basic equations for x and y

$$x = \sum_{i=1}^{N_e} x_i \varphi_i(\xi, \eta) \qquad y = \sum_{i=1}^{N_e} y_i \varphi_i(\xi, \eta)$$

Quadrilateral Transformations IV

- Derivatives of x and y coordinates from definitions in terms of shape functions
- Recall that $\phi_i = \phi_i(\xi,\eta)$
  - Actual equation depends on choice of basis functions (linear, quadratic, etc.)

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{N_e} x_i \frac{\partial \varphi_i}{\partial \xi} \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{N_e} y_i \frac{\partial \varphi_i}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{N_e} x_i \frac{\partial \varphi_i}{\partial \eta} \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{N_e} y_i \frac{\partial \varphi_i}{\partial \eta}$$

Problem With New Coordinates

- Previous slide has equations like

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{N_e} x_i \frac{\partial \varphi_i}{\partial \eta} \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{N_e} y_i \frac{\partial \varphi_i}{\partial \xi}$$

- Can find these from ϕ_i, but need opposite derivatives in transformations

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \frac{\partial g}{\partial y} = \frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial g}{\partial \eta}\frac{\partial \eta}{\partial y}$$

- How do we relate these derivatives?

Coordinate Transformations

- Want to changes in (x,y) to changes in (ξ,η) and *vice versa*

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \quad and \quad dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \quad and \quad d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy$$

- The area relationship is dA = dxdy = Jdξdη
- J = Jacobian determinant $J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$

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Basis for Inverse Derivatives

- Write dx and dy for x = x(ξ,η) and y = y(ξ,η) as a matrix equation
- Also write dξ and δη for ξ = ξ(x,y) and η = η(x,y) as a matrix equation

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \qquad \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

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Basis for Inverse Derivatives II

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \Rightarrow \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

- Comparing these two equations shows that one matrix must be the inverse of the other

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

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Basis for Inverse Derivatives III

- If **B** = **A**$^{-1}$ for a 2 x 2 matrix, $b_{11} = a_{22}/\det(\mathbf{A})$, $b_{12} = -a_{12}/\det(\mathbf{A})$, $b_{21} = -a_{21}/\det(\mathbf{A})$, and $b_{22} = a_{11}/\det(\mathbf{A})$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

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Inverse Derivatives

- Use equations to compute ∂ξ/∂x, ∂ξ/∂y, ∂η/∂x, and ∂η/∂y from ∂x/∂ξ, ∂x/∂η, ∂y/∂ξ, and ∂y/∂η

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \eta} = \frac{y_\eta}{J} \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \eta} = -\frac{x_\eta}{J}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial \xi} = -\frac{y_\xi}{J} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \xi} = \frac{x_\xi}{J}$$

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Quadrilateral Transformations V

$$\frac{\partial \phi_m}{\partial x} = \frac{\partial \phi_m}{\partial \xi}\xi_x + \frac{\partial \phi_m}{\partial \eta}\eta_x = \frac{\partial \phi_m}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_m}{\partial \eta}\frac{y_\xi}{J}$$

$$\frac{\partial \phi_m}{\partial y} = \frac{\partial \phi_m}{\partial \xi}\xi_y + \frac{\partial \phi_m}{\partial \eta}\eta_y = -\frac{\partial \phi_m}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_m}{\partial \eta}\frac{x_\xi}{J}$$

- Apply these results and dxdy = Jdξdη to Galerkin formula for A_{ki}
- Get final integral for A_{ki} coefficients for quadrilateral element

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Quadrilateral Transformations VI

$$A_{ki} = \int_\Omega \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \phi_i \right] dxdy \qquad dxdy = Jd\xi d\eta$$

$$\frac{\partial \phi_i}{\partial x} = \frac{\partial \phi_i}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_i}{\partial \eta}\frac{y_\xi}{J} \qquad \frac{\partial \phi_k}{\partial x} = \frac{\partial \phi_k}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_k}{\partial \eta}\frac{y_\xi}{J}$$

$$\frac{\partial \phi_i}{\partial y} = -\frac{\partial \phi_i}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_i}{\partial \eta}\frac{x_\xi}{J} \qquad \frac{\partial \phi_k}{\partial y} = -\frac{\partial \phi_k}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_k}{\partial \eta}\frac{x_\xi}{J}$$

- Substitute the expressions for space derivatives of ϕ_i and ϕ_k into A_{ki} and simplify

Quadrilateral Transformations VII

$$\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_k}{\partial y} = \left[\frac{\partial \phi_i}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_i}{\partial \eta}\frac{y_\xi}{J}\right]\left[\frac{\partial \phi_k}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_k}{\partial \eta}\frac{y_\xi}{J}\right] + \left[-\frac{\partial \phi_i}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_i}{\partial \eta}\frac{x_\xi}{J}\right]\left[-\frac{\partial \phi_k}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_k}{\partial \eta}\frac{x_\xi}{J}\right]$$

$$= \frac{\partial \phi_i}{\partial \xi}\frac{y_\eta}{J}\frac{\partial \phi_k}{\partial \xi}\frac{y_\eta}{J} - \frac{\partial \phi_i}{\partial \xi}\frac{y_\eta}{J}\frac{\partial \phi_k}{\partial \eta}\frac{y_\xi}{J} - \frac{\partial \phi_i}{\partial \eta}\frac{y_\xi}{J}\frac{\partial \phi_k}{\partial \xi}\frac{y_\eta}{J} + \frac{\partial \phi_i}{\partial \eta}\frac{y_\xi}{J}\frac{\partial \phi_k}{\partial \eta}\frac{y_\xi}{J}$$

$$+ \frac{\partial \phi_i}{\partial \xi}\frac{x_\eta}{J}\frac{\partial \phi_k}{\partial \xi}\frac{x_\eta}{J} - \frac{\partial \phi_i}{\partial \xi}\frac{x_\eta}{J}\frac{\partial \phi_k}{\partial \eta}\frac{x_\xi}{J} - \frac{\partial \phi_i}{\partial \eta}\frac{x_\xi}{J}\frac{\partial \phi_k}{\partial \xi}\frac{x_\eta}{J} + \frac{\partial \phi_i}{\partial \eta}\frac{x_\xi}{J}\frac{\partial \phi_k}{\partial \eta}\frac{x_\xi}{J}$$

$$= \frac{\partial \phi_i}{\partial \xi}\left[\frac{\partial \phi_k}{\partial \xi}\frac{x_\eta^2 + y_\eta^2}{J^2} - \frac{\partial \phi_k}{\partial \eta}\frac{y_\eta y_\xi + x_\eta x_\xi}{J^2}\right]$$

Quadrilateral Transformations VIII

$$A_{ki} = \int_\Omega \left[\frac{\partial \phi_i}{\partial x}\frac{\partial \varphi_k}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \varphi_k}{\partial y} - \varphi_k a^2 \phi_i \right] dxdy$$

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \xi}\left[\frac{x_\eta^2 + y_\eta^2}{J^2}\frac{\partial \varphi_k}{\partial \xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J^2}\frac{\partial \varphi_i}{\partial \eta}\right] J d\xi d\eta +$$

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \eta}\left[\frac{x_\xi^2 + y_\xi^2}{J^2}\frac{\partial \varphi_k}{\partial \eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J^2}\frac{\partial \varphi_k}{\partial \xi}\right] J d\xi d\eta$$

$$-\int_{-1}^{1}\int_{-1}^{1} \varphi_k a^2 \phi_i J d\xi d\eta$$

Quadrilateral Transformations IX

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \xi}\left[\frac{x_\eta^2 + y_\eta^2}{J}\frac{\partial \varphi_k}{\partial \xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_i}{\partial \eta}\right] d\xi d\eta +$$

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \eta}\left[\frac{x_\xi^2 + y_\xi^2}{J}\frac{\partial \varphi_k}{\partial \eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_k}{\partial \xi}\right] d\xi d\eta$$

$$-\int_{-1}^{1}\int_{-1}^{1} \varphi_k a^2 \phi_i J d\xi d\eta$$

- Derivatives in these integrals can be found directly from the shape functions $\phi(\xi,\eta)$

Evaluating A_{ki} Integrals

- Use numerical integration algorithm known as Gauss Quadrature
  - Uses fixed points to obtain high accuracy with a small number of points

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n} \gamma_j f(x_j)$$

  - Choose n and find weights, γ_j, and sample points, x_i, in tables
  - Formula with n sample points, x_i , is exact for a polynomial of order 2n – 1 (or less)

Gauss Quadrature Data

- Sample points, x_i, and weights, γ_i, for small values of n typical of those used in finite-element programs

n	x_i	γ_i	n	x_i	γ_i
1	0	2	4	±0.861136312	0.347854845
2	$\pm 1/3^{1/2}$	1		±0.339981044	0.652145155
3	$\pm(0.6)^{1/2}$	5/9	5	±0.906179846	0.236926885
	0	8/9		±0.538469310	0.478628671
				0	0.568888889

Element Gauss Quadrature

- Apply quadrature rule in two dimensions

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx \sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_k\gamma_j f(\xi_k,\eta_j)$$

- Finite-element quadrilateral integrating over ξ and η has n^2 points in element
- Example for n = 2

$$\xi_1 = -1/\sqrt{3} \quad \gamma_1 = 1 \quad \eta_1 = -1/\sqrt{3} \quad \gamma_1 = 1$$
$$\xi_2 = 1/\sqrt{3} \quad \gamma_2 = 1 \quad \eta_1 = -1/\sqrt{3} \quad \gamma_1 = 1$$
$$\xi_1 = -1/\sqrt{3} \quad \gamma_1 = 1 \quad \eta_2 = 1/\sqrt{3} \quad \gamma_2 = 1$$
$$\xi_2 = 1/\sqrt{3} \quad \gamma_2 = 1 \quad \eta_2 = 1/\sqrt{3} \quad \gamma_2 = 1$$

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Gauss Quadrature for A_{kj}

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

$$f(\xi,\eta) = \frac{\partial\varphi_i}{\partial\xi}\left[\frac{x_\eta^2+y_\eta^2}{J}\frac{\partial\varphi_k}{\partial\xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial\varphi_i}{\partial\eta}\right] +$$

$$\frac{\partial\varphi_i}{\partial\eta}\left[\frac{x_\xi^2+y_\xi^2}{J}\frac{\partial\varphi_k}{\partial\eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial\varphi_k}{\partial\xi}\right] - \varphi_k a^2 \phi_i J$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx \sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_k\gamma_j f(\xi_k,\eta_j)$$

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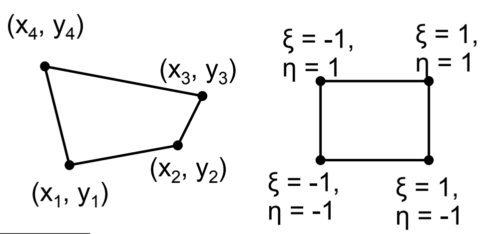
Quadrilateral with Linear ϕ_i



Note: $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$
$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

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Shape Functions Derivatives

- Apply equations for derivatives to linear shape functions, ϕ_i

$$\frac{\partial x}{\partial\xi} = \sum_{i=1}^{N_e} x_i\frac{\partial\varphi_i}{\partial\xi} \qquad \frac{\partial y}{\partial\xi} = \sum_{i=1}^{N_e} y_i\frac{\partial\varphi_i}{\partial\xi}$$
$$\frac{\partial x}{\partial\eta} = \sum_{i=1}^{N_e} x_i\frac{\partial\varphi_i}{\partial\eta} \qquad \frac{\partial y}{\partial\eta} = \sum_{i=1}^{N_e} y_i\frac{\partial\varphi_i}{\partial\eta}$$
$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$
$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

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Shape Functions Derivatives II

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \quad \frac{\partial\varphi_1}{\partial\xi} = -\frac{(1-\eta)}{4} \quad \frac{\partial\varphi_1}{\partial\eta} = -\frac{(1-\xi)}{4}$$
$$\varphi_2 = \frac{(1+\xi)(1-\eta)}{4} \quad \frac{\partial\varphi_2}{\partial\xi} = \frac{(1-\eta)}{4} \quad \frac{\partial\varphi_2}{\partial\eta} = -\frac{(1+\xi)}{4}$$
$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \quad \frac{\partial\varphi_3}{\partial\xi} = \frac{(1+\eta)}{4} \quad \frac{\partial\varphi_3}{\partial\eta} = \frac{(1+\xi)}{4}$$
$$\varphi_4 = \frac{(1-\xi)(1+\eta)}{4} \quad \frac{\partial\varphi_4}{\partial\xi} = -\frac{(1+\eta)}{4} \quad \frac{\partial\varphi_4}{\partial\eta} = \frac{(1-\xi)}{4}$$

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Coordinate Derivatives – x

$$\frac{\partial x}{\partial\xi} = \sum_{i=1}^{4} x_i\frac{\partial\varphi_i}{\partial\xi} = -\frac{(1-\eta)}{4}x_1 + x_2\frac{(1-\eta)}{4} + x_3\frac{(1+\eta)}{4}$$
$$-x_4\frac{(1+\eta)}{4} = \frac{1-\eta}{4}(x_2 - x_1) + \frac{1+\eta}{4}(x_3 - x_4)$$
$$\frac{\partial x}{\partial\eta} = \sum_{i=1}^{4} x_i\frac{\partial\varphi_i}{\partial\eta} = -x_1\frac{(1-\xi)}{4} - x_2\frac{(1+\xi)}{4} + x_3\frac{(1+\xi)}{4}$$
$$+x_4\frac{(1-\xi)}{4} = \frac{1-\xi}{4}(x_4 - x_1) + \frac{1+\xi}{4}(x_3 - x_2)$$

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Coordinate Derivatives – y

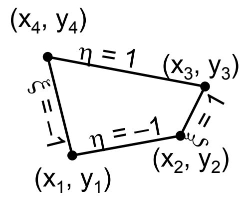
$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} y_i \frac{\partial \varphi_i}{\partial \xi} = -\frac{(1-\eta)}{4} y_1 + y_2 \frac{(1-\eta)}{4} + y_3 \frac{(1+\eta)}{4}$$

$$- y_4 \frac{(1+\eta)}{4} = \frac{1-\eta}{4}(y_2 - y_1) + \frac{1+\eta}{4}(y_3 - y_4)$$

$$\frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} y_i \frac{\partial \varphi_i}{\partial \eta} = -y_1 \frac{(1-\xi)}{4} - y_2 \frac{(1+\xi)}{4} + y_3 \frac{(1+\xi)}{4}$$

$$+ y_4 \frac{(1-\xi)}{4} = \frac{1-\xi}{4}(y_4 - y_1) + \frac{1+\xi}{4}(y_3 - y_2)$$

Quadrilateral Derivatives



$$x_\xi = \frac{1-\eta}{4}(x_2 - x_1) + \frac{1+\eta}{4}(x_3 - x_4)$$

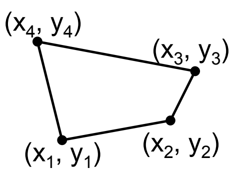
$$y_\xi = \frac{1-\eta}{4}(y_2 - y_1) + \frac{1+\eta}{4}(y_3 - y_4)$$

$$x_\eta = \frac{1-\xi}{4}(x_4 - x_1) + \frac{1+\xi}{4}(x_3 - x_2)$$

$$y_\eta = \frac{1-\xi}{4}(y_4 - y_1) + \frac{1+\xi}{4}(y_3 - y_2)$$

- For rectangle ($x_1 = x_4$, $x_2 = x_3$, $y_1 = y_2$, and $y_3 = y_4$) $x_\eta = y_\xi = 0$ and $x_\xi = \Delta x/2$ and $y_\eta = \Delta y/2$

From Global to Element



- Global equation for nodal values defined for entire region
- Has boundary term that is present only for boundary elements

$$\sum_{i=0}^{N} A_{ki} u_i = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds \quad k = 0, \ldots, N$$

- Apply to individual element with only four nodes

Elements without Boundaries

- Global equation includes external boundary term

$$\sum_{i=0}^{N} A_{ki} u_i = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds \quad k = 0, \ldots, N$$

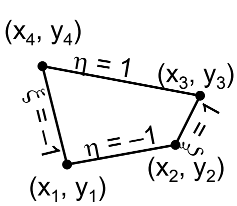
- Term does not exist for element with no nodes on external boundary

$$\sum_{i=1}^{4} A_{ki} u_i = 0 \quad k = 1, \ldots, 4$$

$$A_{k1}u_1 + A_{k2}u_2 + A_{k3}u_3 + A_{k4}u_4 = 0 \quad k = 1, \ldots, 4$$

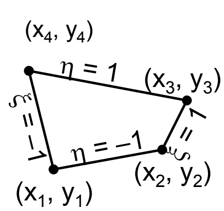
- Add boundary term if element contains an external boundary

Boundary Terms



- Element boundaries lie along line of constant ξ = ±1 or constant η = ±1
- Boundary integral is found along these lines
- Two cases to consider
  - Have gradient (2nd or 3rd kind) boundary condition to include in solution
  - Compute gradients from solution for Dirichlet boundary condition

Boundary Integral

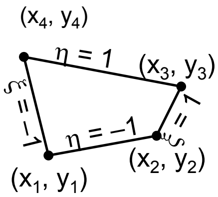


- Find integral below whenever one side of element is on external boundary

$$\int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds \approx \left(\frac{\partial u}{\partial n}\right)_{side} \int_{start}^{end} \varphi_k ds$$

- Differential distance, ds = $S_{\xi=\pm1} d\eta/2$ or $S_{\eta=\pm1} d\xi/2$ where S is length of side
  - *E.g.* $S_{\xi=1} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

ξ = 1 Boundary as Example



$$\boxed{\int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds} \qquad S_{\xi=1} = \int_2^3 ds = \int_{-1}^{1} S_{\xi=1} \frac{d\eta}{2}$$

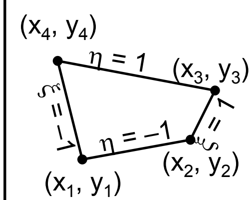
- ϕ_1 and ϕ_4 are zero along $\xi = 1$ boundary
- Length of side, $S_{\xi=1}$, is

$$S_{\xi=1} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

- Differential distance, $ds = S_{\xi=1} d\eta/2$

$$\varphi_2 = \frac{(1+\xi)(1-\eta)}{4} \qquad \varphi_3 = \frac{(1+\xi)(1+\eta)}{4}$$

ξ = 1 Boundary Example II



- Evaluate integral for ϕ_2 and ϕ_3 at $\xi = 1$ $\quad \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds$

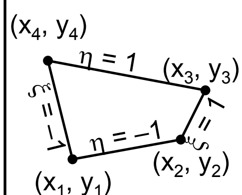
$$(\varphi_2)_{\xi=1} = \frac{(1+1)(1-\eta)}{4} = \frac{(1-\eta)}{2}$$

$$\int_\Gamma \varphi_2 \frac{\partial \hat{u}}{\partial n} ds = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} S_{\xi=1} \int_{-1}^{1} \frac{1-\eta}{2}\frac{d\eta}{2} = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} \frac{S_{\xi=1}}{2}$$

= 1/2

- Same result for ϕ_3

ξ = 1 Boundary Equations

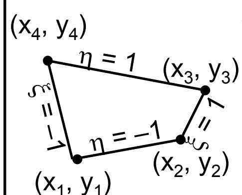


- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} \frac{S_{\xi=1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = 0$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = B$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = B$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = 0$$

- Equations with B not needed for Dirichlet boundary conditions
  - Used after solution to compute gradients

η = 1 Boundary Equations

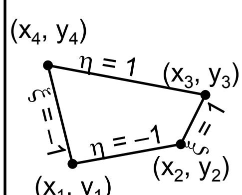


- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\eta=1} \frac{S_{\eta=1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = 0$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = 0$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = B$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = B$$

- Equations with B not needed for Dirichlet boundary conditions
  - Used after solution to compute gradients

ξ = –1 Boundary Equations

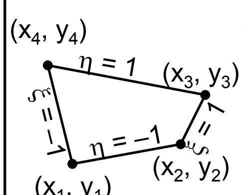


- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\xi=-1} \frac{S_{\xi=-1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = B$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = 0$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = 0$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = B$$

- Same sign on B as in $\xi = 1$ term
  - Outward facing normal derivative is in opposite direction

η = –1 Boundary Equations



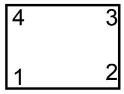
- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\eta=-1} \frac{S_{\eta=-1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = B$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = B$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = 0$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = 0$$

- Same sign on B as in $\eta = 1$ term
  - Outward facing normal derivative is in opposite direction

Assembly Process

- We analyzed element with local numbering system shown at right



- Assembly process accounts for fact that nodes on one element are also present on other elements
  - Look at all occurrences of each node on all elements it shares
  - Add all element equations for node where it is "central" node (multiplied by A_{kk})

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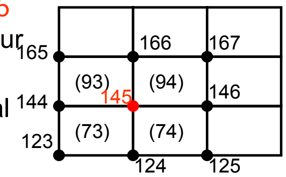
Assembly Process

- Typical node (145 here) is part of four elements
- Grid shows global node and (element) numbers



- Local and global indices for each element

	1	2	3	4
(73)	123	124	145	144
(74)	124	125	146	145
(93)	144	145	166	165
(94)	145	146	167	166

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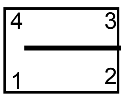
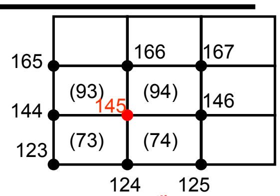
Global Indices



- Node 145 element equations



$$A_{31}^{(73)}u_{123} + A_{32}^{(73)}u_{124} + A_{33}^{(73)}u_{145} + A_{34}^{(73)}u_{144} = 0$$
$$A_{41}^{(74)}u_{124} + A_{42}^{(74)}u_{125} + A_{43}^{(74)}u_{146} + A_{44}^{(74)}u_{145} = 0$$
$$A_{21}^{(93)}u_{144} + A_{22}^{(93)}u_{145} + A_{23}^{(93)}u_{166} + A_{24}^{(93)}u_{165} = 0$$
$$A_{11}^{(94)}u_{145} + A_{12}^{(94)}u_{146} + A_{14}^{(94)}u_{166} + A_{13}^{(94)}u_{167} = 0$$

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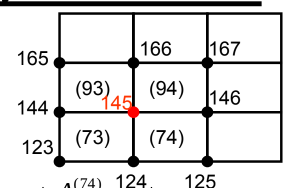
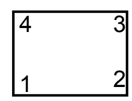
Assembly Process

- Add all element equations with global indices for node 145



$$A_{31}^{(73)}u_{123} + \left[A_{32}^{(73)} + A_{41}^{(74)}\right]u_{124} + A_{42}^{(74)}u_{125} + \left[A_{34}^{(73)} + A_{21}^{(93)}\right]u_{144} + \left[A_{11}^{(94)} + A_{22}^{(93)} + A_{33}^{(73)} + A_{44}^{(74)}\right]u_{145} + \left[A_{12}^{(94)} + A_{43}^{(74)}\right]u_{146} + A_{24}^{(93)}u_{165} + \left[A_{14}^{(94)} + A_{23}^{(93)}\right]u_{166} + A_{13}^{(94)}u_{167} = 0$$



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Uniform Grid Δx = Δy = h

- Here $x_\xi = y_\eta = h/2$, $x_\eta = y_\xi = 0$ and $J = x_\xi y_\eta - x_\eta y_\xi = h^2/4$ giving A_{ki} equations as

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \xi}\left[\frac{x_\eta^2 + y_\eta^2}{J}\frac{\partial \varphi_k}{\partial \xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_i}{\partial \eta}\right] d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \eta}\left[\frac{x_\xi^2 + y_\xi^2}{J}\frac{\partial \varphi_k}{\partial \eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_k}{\partial \xi}\right] d\xi d\eta - \int_{-1}^{1}\int_{-1}^{1} \varphi_k a^2 \phi_i J d\xi d\eta$$

Both are one

Both are zero

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Uniform Grid Δx = Δy = h II

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial \varphi_i}{\partial \xi}\frac{\partial \varphi_k}{\partial \xi} + \frac{\partial \varphi_i}{\partial \eta}\frac{\partial \varphi_k}{\partial \eta} - \frac{\varphi_k (ah)^2 \phi_i}{4}\right] d\xi d\eta$$

- Integration details at end of presentation

$$\mathbf{A} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} - \frac{a^2h^2}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

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Uniform Grid Assembled

- Assemble element equations using finite difference notation for uniform grid

$$4\left(\frac{2}{3}-\frac{a^2h^2}{9}\right)u_{ij}+\left(-\frac{1}{3}-\frac{a^2h^2}{36}\right)\left(u_{i+1j+1}+u_{i+1j-1}+u_{i-1j+1}+u_{i-1j-1}\right)$$
$$+2\left(-\frac{1}{6}-\frac{a^2h^2}{18}\right)\left(u_{i+1j}+u_{ij+1}+u_{i-1j}+u_{ij-1}\right)=0$$
$$\left(8-\frac{4a^2h^2}{3}\right)u_{ij}+\left(-1-\frac{a^2h^2}{12}\right)\left(u_{i+1j+1}+u_{i+1j-1}+u_{i-1j+1}+u_{i-1j-1}\right)$$
$$+\left(-1-\frac{a^2h^2}{3}\right)\left(u_{i+1j}+u_{ij+1}+u_{i-1j}+u_{ij-1}\right)=0$$

Second Order Error Results

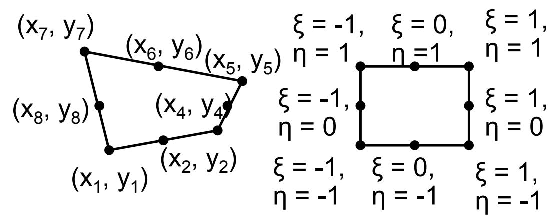
Step size h	Sine BC		Constant BC	
	Maximum error	RMS error	Maximum error	RMS error
0.25	0.0187	0.00723	0.0183	0.00746
0.125	0.00446	0.00195	0.0109	0.00351
0.0625	0.00117	0.000511	0.00947	0.00165
0.03125	0.000278	0.000131	0.00945	0.000799
0.015625	0.0000697	0.0000333	0.00943	0.000393
0.0078125	0.0000175	0.0000084	0.00943	0.000195

- FEM error 20-30% higher for sine BC

Quadratic ϕ_i Quadrilateral

- Can use higher order shape functions
- Will not show complete example here, but will illustrate arrangement
- Have eight nodes on an element
  - Four on corners plus four at midpoints of sides
- Shape functions build in a manner similar to linear shape functions: products of terms in ξ and η

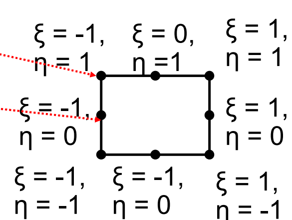
Quadratic ϕ_i Quadrilateral



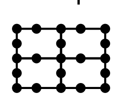
$$\varphi_1=\frac{\xi(\xi-1)\eta(\eta-1)}{4}\quad \varphi_2=\frac{\left(1-\xi^2\right)\eta(\eta-1)}{2}\quad \varphi_3=\frac{\xi(\xi+1)\eta(\eta-1)}{4}$$
$$\varphi_4=\frac{\xi(\xi+1)\left(1-\eta^2\right)}{2}\quad \varphi_5=\frac{\xi(\xi+1)\eta(\eta+1)}{4}\quad \varphi_6=\frac{\left(1-\xi^2\right)\eta(\eta+1)}{2}$$

Quadratic ϕ_i Quadrilateral II

$$\varphi_7=\frac{\xi(\xi-1)\eta(\eta+1)}{4}$$
$$\varphi_8=\frac{\xi(\xi-1)\left(1-\eta^2\right)}{2}$$



- Eight equations for each element
- Each assembled equation away from boundaries will link 4 elements with 21 nodes (21 terms in equation)



Conclusions

- FEM allows easier consideration of arbitrary geometry
- Equations for quadrilaterals use nine nodes to get second-order error
  - Finite-difference compact formula gets fourth-order error with these nodes
- Have used one element type and one shape (basis) function, a linear one
  - Consider other elements and shape functions next week

Additional Materials

- Charts 73 to 79 show the details of the integration of A_{ki} for the case where there is a rectangular grid
- This gives the details of the result shown on slide 66

$$\mathbf{A} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} - \frac{a^2h^2}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

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Shape Function Integrals

- We can write all shape functions as

$$\varphi_i = \frac{(1+a_i\xi)(1+b_i\eta)}{4}$$

$$\frac{\partial\varphi_i}{\partial\xi} = \frac{a_i(1+b_i\eta)}{4}$$

$$\frac{\partial\varphi_i}{\partial\eta} = \frac{b_i(1+a_i\xi)}{4}$$

i	a_i	b_i
1	–1	–1
2	1	–1
3	1	1
4	–1	1

- Use these forms to get integrals of $\phi_i\phi_k$ products

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Shape Function Integrals II

- Split A_{ki} into three integrals
  - Evaluate separately and combine results

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial\varphi_i}{\partial\xi}\frac{\partial\varphi_k}{\partial\xi} + \frac{\partial\varphi_i}{\partial\eta}\frac{\partial\varphi_k}{\partial\eta} - \frac{\varphi_k(ah)^2\phi_i}{4}\right]d\xi d\eta$$

$$I_1 = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\varphi_i}{\partial\xi}\frac{\partial\varphi_k}{\partial\xi}d\xi d\eta \qquad I_2 = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\varphi_i}{\partial\eta}\frac{\partial\varphi_k}{\partial\eta}d\xi d\eta$$

$$I_3 = \int_{-1}^{1}\int_{-1}^{1}\varphi_k\phi_i d\xi d\eta \qquad A_{ki} = I_1 + I_2 - \frac{(ah)^2 I_3}{4}$$

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Shape Function Integrals III

$$I_1 = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\varphi_i}{\partial\xi}\frac{\partial\varphi_k}{\partial\xi}d\eta d\xi = \int_{-1}^{1}\int_{-1}^{1}\frac{a_i(1+b_i\eta)}{4}\frac{a_k(1+b_k\eta)}{4}d\eta d\xi$$

$$= \left[\int_{-1}^{1}d\xi\right]\frac{a_ia_k}{16}\int_{-1}^{1}\left[1+(b_i+b_k)\eta + b_ib_k\eta^2\right]d\eta$$

$$= 2\frac{a_ia_k}{16}\left[\eta + (b_i+b_k)\frac{\eta^2}{2} + b_ib_k\frac{\eta^3}{3}\right]_{-1}^{1}$$

$$= \frac{a_ia_k}{8}\left[2+0+b_ib_k\frac{2}{3}\right] = \frac{a_ia_k}{4} + \frac{a_ia_kb_ib_k}{12}$$

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Shape Function Integrals IV

$$I_2 = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\varphi_i}{\partial\eta}\frac{\partial\varphi_k}{\partial\eta}d\eta d\xi = \int_{-1}^{1}\int_{-1}^{1}\frac{b_i(1+a_i\xi)}{4}\frac{b_k(1+a_k\xi)}{4}d\eta d\xi$$

$$= \left[\int_{-1}^{1}d\eta\right]\frac{b_ib_k}{16}\int_{-1}^{1}\left[1+(a_i+a_k)\xi + a_ia_k\xi^2\right]d\xi$$

$$= 2\frac{b_ib_k}{16}\left[\xi + (a_i+a_k)\frac{\xi^2}{2} + a_ia_k\frac{\xi^3}{3}\right]_{-1}^{1}$$

$$= \frac{b_ib_k}{8}\left[2+0+a_ia_k\frac{2}{3}\right] = \frac{b_ib_k}{4} + \frac{b_ib_ka_ia_k}{12}$$

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Shape Function Integrals V

$$I_3 = \int_{-1}^{1}\int_{-1}^{1}\varphi_i\varphi_k d\eta d\xi = \int_{-1}^{1}\int_{-1}^{1}\frac{(1+a_i\xi)(1+b_i\eta)}{4}\frac{(1+a_k\xi)(1+b_k\eta)}{4}d\eta d\xi$$

$$= \frac{1}{16}\left[\int_{-1}^{1}(1+a_i\xi)(1+a_k\xi)d\xi\right]\int_{-1}^{1}(1+b_i\eta)(1+b_k\eta)d\eta$$

$$= \frac{1}{16}\left[\xi + (a_i+a_k)\frac{\xi^2}{2} + a_ia_k\frac{\xi^3}{3}\right]_{-1}^{1}\left[\eta + (b_i+b_k)\frac{\eta^2}{2} + b_ib_k\frac{\eta^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{16}\left[2+0+a_ia_k\frac{2}{3}\right]\left[2+0+b_ib_k\frac{2}{3}\right] = \frac{1}{4}\left(1+\frac{a_ia_k+b_ib_k}{3}+\frac{a_ia_kb_ib_k}{9}\right)$$

- Next slide gives $A_{ki} = I_1 + I_2 - I_3(ah)^2/4$

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Shape Function Integrals VI

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \xi}\frac{\partial \varphi_k}{\partial \xi} d\eta d\xi = \frac{a_i a_k}{4} + \frac{a_i a_k b_i b_k}{12}$$

$$I_2 = \int_{-1}^{1}\int_{-1}^{1} \frac{\partial \varphi_i}{\partial \eta}\frac{\partial \varphi_k}{\partial \eta} d\eta d\xi = \frac{b_i b_k}{4} + \frac{b_i b_k a_i a_k}{12}$$

$$I_3 = \int_{-1}^{1}\int_{-1}^{1} \varphi_i \varphi_k d\eta d\xi = \frac{1}{4}\left(1 + \frac{a_i a_k + b_i b_k}{3} + \frac{a_i a_k b_i b_k}{9}\right)$$

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial \varphi_i}{\partial \xi}\frac{\partial \varphi_k}{\partial \xi} + \frac{\partial \varphi_i}{\partial \eta}\frac{\partial \varphi_k}{\partial \eta} - \frac{\varphi_k (ah)^2 \phi_i}{4}\right] d\xi d\eta$$

$$= \frac{a_i a_k + b_i b_k}{4} + \frac{a_i a_k b_i b_k}{6} - \frac{(ah)^2}{16}\left(1 + \frac{a_i a_k + b_i b_k}{3} + \frac{a_i a_k b_i b_k}{9}\right)$$

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Shape Function Integrals VII

- Substitute values for a_i, b_i, a_k, and b_k from chart 73
  - Done for A_{11}, ($a_1 = -1$ and $b_1 = -1$) below
  - Next slide gives all A_{ki} as matrices

$$A_{ki} = \frac{a_i a_k + b_i b_k}{4} + \frac{a_i a_k b_i b_k}{6} - \frac{(ah)^2}{16}\left(1 + \frac{a_i a_k + b_i b_k}{3} + \frac{a_i a_k b_i b_k}{9}\right)$$

$$= \frac{(-1)(-1)+(-1)(-1)}{4} + \frac{(-1)(-1)(-1)(-1)}{6} - \frac{(ah)^2}{16}\left(1 + \frac{(-1)(-1)+(-1)(-1)}{3} + \frac{(-1)(-1)}{9}\right)$$

$$= \frac{2}{4} + \frac{1}{6} - \frac{(ah)^2}{16}\left(1 + \frac{2}{3} + \frac{1}{9}\right) = \frac{2}{3} - \frac{1(ah)^2}{9}$$

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Shape Function Integrals VIII

- Result of substitution into all A_{ki}

$$\mathbf{A} = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} - \frac{a^2h^2}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

- Matrices are symmetric so $A_{ik} = A_{ki}$
- Have four basic terms: same node, diagonal opposite and horizontal/vertical

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Shape Function Integrals IX

- Central node

$$A_{kk} = \frac{2}{3} - \frac{a^2h^2}{9}$$

- Diagonally opposite

$$A_{ki} = -\frac{1}{3} - \frac{a^2h^2}{36}$$

- Horizontal or vertical neighbor

$$A_{ki} = -\frac{1}{6} - \frac{a^2h^2}{18}$$

- Assembly process counts central node four times, diagonal opposites once each, and horizontal/vertical neighbor twice

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Shape Function Integrals X

- Assembly on uniform grid using finite difference notation
  - Central node (i, j) counts four times
  - Diagonal opposites (i±1, j±1) count once
  - Horizontal neighbor (i±1, j) count two times
  - Vertical neighbors (i, j±1) count two times

$$4\left(\frac{2}{3} - \frac{a^2h^2}{9}\right)u_{ij} + \left(-\frac{1}{3} - \frac{a^2h^2}{36}\right)\left(u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1}\right.$$

$$\left. + 2\left(-\frac{1}{6} - \frac{a^2h^2}{18}\right)\left(u_{i+1j} + u_{ij+1} + u_{i-1j} + u_{ij-1}\right) = 0\right.$$

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