Introduction to Finite Elements

Larry Caretto
Mechanical Engineering 501AB

Seminar in Engineering Analysis

April 13, 2009

California State University Northridge

Outline

- Review midterm
- Introduction to finite elements
- Basic approaches to finite elements
  - Will start material originally scheduled for April 22
  - Parallel reading for this week is pages 711 to 739 in Hoffman
- Example application in one dimension

California State University Northridge 2

Midterm Results

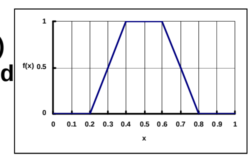
- Number of students = 12
- Maximum possible = 100
- Mean = 45.1
- Median = 43.5
- Standard deviation = 28.2
- Grade distribution

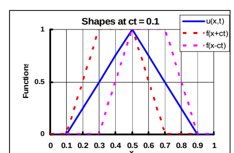
  2 10 14 26 35 40
  47 65 71 73 75 83

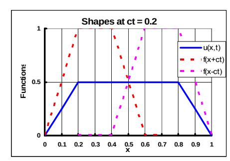
California State University Northridge 3

Problem One

- **Sketch the wave equation solution, u(x,t) at times ct = 0.1, 0.2, and 0.3 for f(x) = u(x,0) as shown at the right.**
  - u(x,t) = [f(x+ct) + f(x-ct)]/2



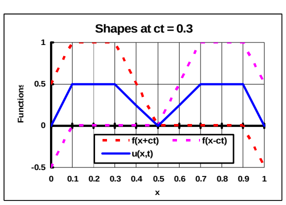




California State University Northridge 4

Problem One II

- For ct = 0.3 must account for periodic extensions of initial conditions for x < 0 and x > 1 entering region 0 ≤ x ≤ 1
  - These are inverse of original initial condition



California State University Northridge 5

Problem Two

- **Outline solution for potential u(x,y,t)**

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad 0 \le x \le L, 0 \le y \le H, t \ge 0 \quad u(x,y,0) = U_0$$

$$u(0,y,t) = 0 \quad u(L,y,t) = U_1 \quad u(x,0,t) = 0 \quad u(x,H,t) = U_2$$

- u(x,y,t) = v(x,y,t) + w(x,y)
  - v(x,y,t) is diffusion equation solution with homogenous boundary conditions
  - w(x,y) is Laplace equation solution requiring superposition: w(x,y) = w_1(x,y) + w_2(x,y)
    - w_1(x,y) and w_2(x,y) each have only one nonhomogenous boundary condition

California State University Northridge 6

Problem Two II

- Solutions in notes

$$v(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$w_1(x,y)=\frac{4U_2}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- Solution for $w_2(x,y)$ found by substituting x and y, L and H (and U_1 for U_2) in w_1 solution

Northridge 7

Problem Two II

$$w_2(x,y)=\frac{4U_1}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi y}{H}\right)\sinh\left(\frac{(2n+1)\pi x}{H}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi L}{H}\right)}$$

- Need to use solutions for w(x,y) to find coefficients in v(x,y,t) solution

$$C_{pq}=\frac{4}{HL}\int_0^H\int_0^L\left[U_0-w_1(x,y)-w_2(x,y)\right]\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

Northridge 8

Problem Three

- **Replace each f_k in equation below by its Taylor series to order of the error**

$$f_i'=\frac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12h}$$

$$f_{i+2}=f_i+f_i'2h+\frac{f_i''(2h)^2}{2!}+\frac{f_i'''(2h)^3}{3!}+\frac{f_i''''(2h)^4}{4!}+\frac{f_i'''''(2h)^5}{5!}+\ldots..$$

$$f_{i+1}=f_i+f_i'h+\frac{f_i''(h)^2}{2!}+\frac{f_i'''(h)^3}{3!}+\frac{f_i''''(h)^4}{4!}+\frac{f_i'''''(h)^5}{5!}+\ldots..$$

$$f_{i-1}=f_i-f_i'h+\frac{f_i''(-h)^2}{2!}+\frac{f_i'''(-h)^3}{3!}+\frac{f_i''''(-h)^4}{4!}+\frac{f_i'''''(-h)^5}{5!}+\ldots..$$

$$f_{i-2}=f_i+f_i'(-2h)+\frac{f_i''(-2h)^2}{2!}+\frac{f_i'''(-2h)^3}{3!}+\frac{f_i''''(-2h)^4}{4!}+\frac{f_i'''''(-2h)^5}{5!}+\ldots..$$

Northridge 9

Problem Three II

- After the algebra

$$\frac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12h}=\frac{(-2+8+8-2)f_i'h}{12h}+\frac{(-32+8+8-32)f_i'''''h^5}{5!(12h)}+\cdots=f_i'+O(h^4)$$

- **Get first derivative of e^x at x = 0 for h = 0.1 and h = 0.05, and find the order of the error**

$$f_i'=\frac{-e^{0+2(0.1)}+8e^{0+0.1}-8e^{0-0.1}+e^{0-2(0.1)}}{12(0.1)}=0.999996662696$$

$$f_i'=\frac{-e^{0+2(0.05)}+8e^{0+0.05}-8e^{0-0.05}+e^{0-2(0.05)}}{12(0.05)}=0.99999979160465$$

$$n\approx\frac{\log(\varepsilon_2)-\log(\varepsilon_1)}{\log(h_2)-\log(h_1)}=\frac{\log(2.08395x10^{-7})-\log(3.3373x10^{-6})}{\log(0.05)-\log(0.1)}=4.001$$

Northridge 10

Why Finite Elements

- A different approach for converting differential equations into algebraic equations
  - More complicated approach than finite differences
  - Capable of handling complex geometry more accurately than finite differences
    - Will not cover finite-volume a melding of the two
  - Uses concept of interpolation polynomials
  - Introduces gradients naturally

Northridge 11

Finite Element Methods

- Designed for 2D and 3D geometries
- Can use for 1D case as example
- Basic idea is to divide region into small elements (line, area, volume)
- Use interpolating polynomial for each element
  - Represent both geometry (independent variables) and dependent variable
  - Polynomials called basis functions or shape functions

Northridge 12

Finite Element Methods II

- Start analysis for region
- Look at set of small elements in region
- Assemble analyses for individual elements is into a set of nodal equations for the entire region
  - Result is set of algebraic equations for the dependent variable at nodes that are points on elements
  - Converts differential equations into a set of algebraic equations at distinct points

California State University Northridge 13

Finite Element Analyses

- How do we represent differential equation in terms of polynomials?
  - Variational approach (Rayleigh-Ritz) method formulates problem as the maximum (or minimum) of a function integral
    - Apply directly to problems in solid mechanics governed by a variational principle (Hamilton's principle)
  - Method of weighted residuals
    - Applies to general differential equations
  - Least squares approaches

California State University Northridge 14

Method of Weighted Residuals

- MWR uses shape functions $\phi_i(\mathbf{x})$ or $N_i(x)$ for each node in region to write approximate solution for true u(**x**)

$$\hat{u} = \sum_{i=1}^{N} u_i \phi_i(\mathbf{x})$$

- Have to find all the u_i values
- Write differential equation as L(u) – b = 0
- Weighting functions $w_i(\mathbf{x})$ for individual nodes in region that satisfy this equation

$$\int_\Omega w_i\left[L(\hat{u}) - b\right] d\Omega = 0 \quad i = 1, \ldots N$$

California State University Northridge 15

MWR Example

- Poisson's equation for $0 \le x \le L$ and $0 \le y \le H$ $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -Q$
- Pick N points in region including nodes on the boundary
- Pick a set of weighting functions w_i
  - Choice for w_i determines final form of algebraic equations solved

Example for L(u)

$$\int_0^L \int_0^H w_i \left[\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + Q\right] dy dx = 0 \quad i = 0, \ldots N$$

California State University Northridge 16

What are Weight Functions?

- Different approaches to MWR use different weight functions
- Galerkin's method uses $w_i = \phi_i$
  - Gives same result as variational approach when a variational principle is possible
- Colocation method uses $w_i = \delta(\mathbf{x} - \mathbf{x}_i)$
- Dirac delta function $\delta(\mathbf{x} - \mathbf{x}_i)$ result

$$\int_\Omega f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_i) d\Omega = f(\mathbf{x}_i)$$

California State University Northridge 17

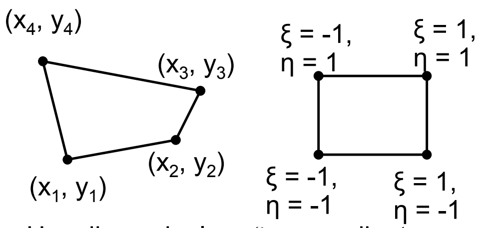
Shape (Basis) Functions

- Simplest shape functions are linear for 1D or bilinear for 2D
- For a linear element between nodes i (at ξ = -1) and i + 1 (at ξ = 1) we have $\phi_i = (1 - \xi)/2$ and $\phi_{i+1} = (1 + \xi)/2$
- $x = x_i \phi_i + x_i \phi_{i+1}$ is correct at nodes
- Bilinear functions for 2D element have the form $(1 \pm \xi)(1 \pm \eta)/2$

$$\xi = \frac{x - \frac{x_1 + x_2}{2}}{\frac{x_2 - x_1}{2}}$$

California State University Northridge 18

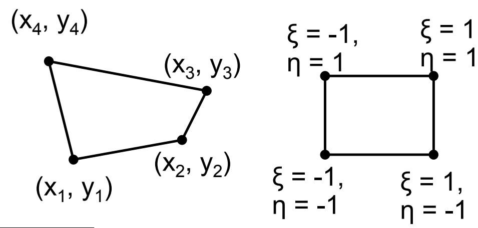
Two-dimensional Element



- Use dimensionless ξ-η coordinate system for shape functions
- Simplest element has one shape function, ϕ_i, at each corner

California State University Northridge 19

Bilinear Basis Functions



Note: $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

California State University Northridge 20

Using Shape Functions

- Model geometry and dependent variable over the element
- Use of same shape functions for both is called isoparametric element
- Shape functions associated with element nodes such that $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$x = \sum_{i=1}^{4} x_i \varphi_i \qquad y = \sum_{i=1}^{4} y_i \varphi_i \qquad \hat{u} = \sum_{i=1}^{4} u_i \varphi_i$$

California State University Northridge 21

Modeling Differential Equation

- Look at simple one-dimensional example: $d^2u/dx^2 + a^2u = 0$
- Equation for u in terms of shape functions gives approximate value $\hat{u} = \sum_{i=1}^{N} u_i \varphi_i$
- Seek solution in which differential equation is satisfied in an average way over the region; w_i is weighting function

$$\int_0^L w_i(x)\left[\frac{d^2\hat{u}}{dx^2} + a^2\hat{u}\right]dx = 0 \qquad i = 0, \ldots N$$

California State University Northridge 22

Modeling Differential Equation II

- Last equation on previous slide is general MWR equation for one dimension
- Various choices are used for weighting functions, w_i
- Galerkin method uses $w_i = \phi_i$
- Known to match variational results for certain linear problems

$$\int_0^L \varphi_i(x)\left[\frac{d^2\hat{u}}{dx^2} + a^2\hat{u}\right]dx = 0 \qquad i = 0, \ldots N$$

California State University Northridge 23

Modeling Differential Equation III

- Use integration by parts to eliminate second derivatives which give zero for linear shape functions

$$\int_0^L \varphi_i \frac{d^2\hat{u}}{dx^2}dx = \int_0^L \varphi_i \frac{d}{dx}\left(\frac{d\hat{u}}{dx}\right)dx = \int_0^L \underbrace{\varphi_i}_{u}\, \underbrace{d\left(\frac{d\hat{u}}{dx}\right)}_{dv}$$

$$= \left[\varphi_i \frac{d\hat{u}}{dx}\right]_0^L - \int_0^L \frac{d\hat{u}}{dx} d\varphi_i = \left[\varphi_i \frac{d\hat{u}}{dx}\right]_0^L - \int_0^L \frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx}dx$$

- Remember there are N+1 of these equations where i = 0, …, N

California State University Northridge 24

Modeling Differential Equation IV

- With result of integration by parts, the original equation is (for each i = 0, …., N)

$$\int_0^L \varphi_i \left[\frac{d^2\hat{u}}{dx^2} + a^2\hat{u} \right] dx = \left[\varphi_i \frac{d\hat{u}}{dx} \right]_0^L - \int_0^L \frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} dx$$

$$+ \int_0^L \varphi_i a^2 \hat{u}\, dx = \left[\varphi_i \frac{d\hat{u}}{dx} \right]_0^L - \int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u} \right] dx = 0$$

- Next steps: handle boundary terms, separate into elements, and introduce shape functions for u

California State University Northridge 25

Modeling Differential Equation V

- At x = 0, $\phi_0 = 1$ and all other $\phi_i = 0$
- At x = L, $\phi_N = 1$ and all other $\phi_i = 0$
- Have special equations for i = 0 and i = N

$$i = 0 \qquad -(1)\left.\frac{d\hat{u}}{dx}\right|_{x=0} - \int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_0}{dx} - \varphi_0 a^2 \hat{u} \right] dx = 0$$

$$i = N \qquad (1)\left.\frac{d\hat{u}}{dx}\right|_{x=L} - \int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_N}{dx} - \varphi_N a^2 \hat{u} \right] dx = 0$$

$$\text{All other } i \qquad \int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u} \right] dx = 0$$

California State University Northridge 26

Modeling Differential Equation VI

$$\int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u} \right] dx = \int_0^L \frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} dx - \int_0^L \varphi_i a^2 \hat{u} dx = 0$$

- Substitute shape function equation for u into integrals $\hat{u} = \sum_{j=0}^{N} u_j \varphi_j$

$$\int_0^L \frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} dx = \int_0^L \frac{d\sum_{j=0}^{N} u_j\varphi_j}{dx}\frac{d\varphi_i}{dx} dx = \sum_{j=0}^{N} u_j \int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} dx$$

$$\int_0^L \varphi_i a^2 \hat{u} dx = \int_0^L \left[\varphi_i a^2 \sum_{j=0}^{N} u_j \varphi_j \right] dx = \sum_{j=0}^{N} u_j \int_0^L \varphi_i a^2 \varphi_j dx$$

California State University Northridge 27

Modeling Differential Equation VII

$$\int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u} \right] dx = \int_0^L \frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} dx - \int_0^L \varphi_i a^2 \hat{u} dx = 0$$

- Result of previous integration is

$$\int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u} \right] dx = \sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx = 0$$

- This is a system of simultaneous linear equations in the unknowns u_j

$$\sum_{j=0}^{N} A_{ij} u_j = \sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx = 0$$

California State University Northridge 28

Modeling Summary

- For the ODE u'' + a²u = 0
  - Get a grid of nodes, x_i
  - Each node has a shape function, $\phi_i(x)$ or $N_i(x)$ with the property that $f_i(x_j) = \delta_{ij}$
  - We want to find the value u_i at each node
  - The algebraic equations to do this are

$$\sum_{j=0}^{N} u_j = 0 \quad where \quad A_{ij} = \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx$$

  - Choose ϕ_i and evaluate A_{ij}
    - Linear equations for ϕ_i are easiest

California State University Northridge 29

Linear Shape (Basis) Functions

$$\varphi_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

Get needed shape functions by substituting i-1 and i +1 for i

Substitute shape functions and derivatives into integrals

$$\frac{d\varphi_i(x)}{dx} = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{1}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{-1}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

California State University Northridge 30

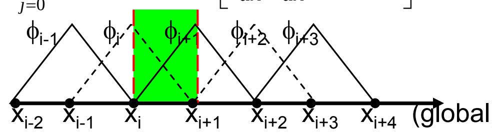
Introduction to Elements

- Note that all shape functions are defined for entire region, but are zero for all but a small part of the region
  - Most A_{ij} coefficients will be zero
- In the 1D case we have integral from 0 to L, but basis function, f_i, is nonzero only between x_{i-1} and x_{i+1}
- Look at element between x_i and x_{i+1} where there are only two nonzero element basis functions

Element Analysis

- Same analysis for individual elements

$$\sum_{j=0}^{N} A_{ij}u_j \qquad A_{ij} = \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx$$

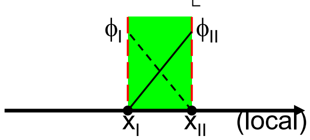


- Look at one typical element where there are only two nonzero shape functions

Element Shape Functions

- Use element numbering scheme

$$\sum_{j=I}^{II} A_{ij}u_j = 0 \qquad A_{ij} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx \quad i = I, II$$



$$\varphi_I(x) = \frac{x_{II} - x}{x_{II} - x_I}$$

$$\varphi_{II}(x) = \frac{x - x_I}{x_{II} - x_I}$$

- Have two equations for element: $A_{I,I}u_I + A_{I,II}u_{II} = 0$ and $A_{II,I}u_I + A_{II,II}u_{II} = 0$

One Element Integral

$$\sum_{j=I}^{II} A_{ij}u_j = \sum_{j=I}^{II} u_j \int_{x_I}^{x_{II}} \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = 0 \quad i = I, II$$

$$A_{ij} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx \qquad \varphi_I(x) = \frac{x_{II} - x}{x_{II} - x_I}$$

$$A_{I,II} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_{II}}{dx}\frac{d\varphi_I}{dx} - \varphi_I a^2 \varphi_{II}\right]dx \qquad \varphi_{II}(x) = \frac{x - x_I}{x_{II} - x_I}$$

$$A_{I,II} = \int_{x_I}^{x_{II}} \left[\frac{1}{x_{II} - x_I}\left(\frac{-1}{x_{II} - x_I}\right) - \frac{x_{II} - x}{x_{II} - x_I} a^2 \frac{x - x_I}{x_{II} - x_I}\right]dx$$

Element Integral Results

- $A_{m,n}$ coefficients for element use α and β notation to summarize results

$$A_{I,I} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_I}{dx}\frac{d\varphi_I}{dx} - \varphi_I a^2 \varphi_I\right]dx = \alpha_I \qquad \alpha_I = -\frac{1}{x_{II} - x_I}$$

$$A_{I,II} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_{II}}{dx}\frac{d\varphi_I}{dx} - \varphi_I a^2 \varphi_{II}\right]dx = \beta_I \qquad + \frac{a^2}{3}(x_{II} - x_I)$$

$$A_{II,I} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_I}{dx}\frac{d\varphi_{II}}{dx} - \varphi_{II} a^2 \varphi_I\right]dx = \beta_I \qquad \beta_I = \frac{1}{x_{II} - x_I}$$

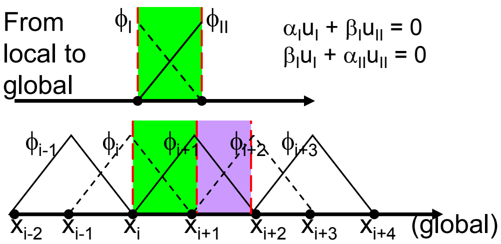
$$A_{II,II} = \int_{x_I}^{x_{II}} \left[\frac{d\varphi_{II}}{dx}\frac{d\varphi_{II}}{dx} - \varphi_{II} a^2 \varphi_{II}\right]dx = \alpha_I \qquad + \frac{a^2}{6}(x_{II} - x_I)$$

Element Assembly

- Return to global numbering scheme
- Two equations for each element
  - $\alpha_I u_I + \beta_I u_{II} = 0$ and $\beta_I u_I + \alpha_{II} u_{II} = 0$
- Weight function i + 1 produces two element equations that are really one global equation
  - Node I in element from i +1 to i + 2
  - Node II in element from i to i + 1

Element Assembly II

- From local to global



- Weight function ϕ_{i+1} is ϕ_{II} in element from x_i to x_{i+1} and ϕ_I from x_{i+1} to x_{i+2}
  - Combine both for ϕ_{i+1} equation

California State University Northridge 37

Element Assembly III

- Combine element equations from some global weight function at each node
- Use node i + 1 as general global node
  - $\alpha_I u_I + \beta_I u_{II} = 0$ from element i + 1 to i + 2 becomes $\alpha_{i+1}u_{i+1} + \beta_{i+1}u_{i+2} = 0$
  - $\beta_I u_I + \alpha_I u_{II} = 0$ from element i to i + 1 becomes $\beta_i u_i + \alpha_i u_{i+1} = 0$
  - Adding both equations gives $\beta_i u_i + (\alpha_i + \alpha_{i+1})u_{i+1} + \beta_{i+1}u_{i+2} = 0$
- In 1D problems, can also use global approach (see supplementary charts)

California State University Northridge 38

Equations to be Solved

$$\alpha_0 u_0 + \beta_0 u_1 = -\left.\frac{du}{dx}\right|_{x=x_0} \qquad \alpha_{N-1}u_{N-1} + \beta_{N-1}u_N = \left.\frac{du}{dx}\right|_{x=x_N}$$

$$\beta_i u_{i-1} + (\alpha_i + \alpha_{i+1})u_i + \beta_{i+1}u_{i+1} = 0 \quad i = 1,\ldots,N-1$$

- Tridiagonal system of N+1 equations with N+3 variables
  - N+1 temperature values and 2 boundary gradients
  - Boundary conditions will specify two other equations

California State University Northridge 39

Boundary Gradients

- If we have Dirichlet boundary conditions, we can solve for temperatures then find gradients
- For Neumann or mixed boundary conditions, we must include gradients in tridiagonal solution
- Write boundary conditions as a du/dx + b u = c and make $g_0 = du/dx|_{x=0}$ the first variable and $g_L = du/dx|_{x=L}$ the last one

California State University Northridge 40

Finite Element Equations

- Equations below only handle boundary conditions with gradients ($a_0 \neq 0$ and $a_N \neq 0$)

$$\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \alpha_0+\alpha_1 & \beta_1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ & \beta_1 & \alpha_1+\alpha_2 & \beta_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \beta_2 & \alpha_2+\alpha_3 & \beta_3 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \ddots & \beta_{N-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \beta_{N-1} & \alpha_{N-1}+\alpha_N & -1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & b_N & a_N \end{bmatrix} \begin{bmatrix} g_0 \\ u_0 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \\ u_N \\ g_N \end{bmatrix} = \begin{bmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ c_N \end{bmatrix}$$

California State University Northridge 41

Solution Errors for a = 2

N	100	100	10	10
Method	FD	FE	FD	FE
e_{RMS}	1.7×10^{-5}	1.7×10^{-5}	1.8×10^{-3}	1.8×10^{-3}
e_{max}	2.4×10^{-5}	2.4×10^{-5}	2.4×10^{-3}	2.4×10^{-3}
$e_{grad}(0)$	3.6×10^{-4}	7.0×10^{-5}	3.6×10^{-2}	7.0×10^{-3}
$e_{grad}(L)$	2.1×10^{-4}	9.6×10^{-5}	1.8×10^{-2}	9.5×10^{-3}

California State University Northridge 42

Solution Errors for a = 0.2

N	100	100	10	10
Method	FD	FE	FD	FE
e_{RMS}	6.2×10^{-10}	6.2×10^{-10}	6.5×10^{-8}	6.5×10^{-8}
e_{max}	8.6×10^{-10}	8.6×10^{-10}	8.5×10^{-8}	8.5×10^{-8}
$e_{grad}(0)$	1.3×10^{-6}	2.2×10^{-9}	1.3×10^{-4}	2.2×10^{-7}
$e_{grad}(L)$	1.3×10^{-6}	4.5×10^{-9}	1.3×10^{-4}	4.4×10^{-7}

43

Notes on the Error

- The formulations used here for finite elements and finite differences have second order error
  - Notes both equations almost the same
- Although temperature errors are similar, finite elements gives smaller errors in the gradients
- The heat source parameter, $a^2 = b/k$, can change the error for a given h

44

Higher Order Shape Functions

- Both finite differences and finite elements can decrease grid size to improve accuracy
- Finite differences can use $O(h)$, $O(h^2)$, $O(h^3)$, etc. expressions for more accuracy with same grid spacing
- Finite elements can use higher-order shape functions to give more accuracy with same element size
- Details on supplemental charts

45

Quadratic Solution Results

- Solve $d^2u/dx^2 + a^2u = 0$ with Dirichlet boundary conditions with a = 2
- 4th order error by log error vs. log h plot

Number of Elements	Element width, h	Maximum Error	RMS Error
5	0.2	3.70E-05	2.40E-05
10	0.1	2.40E-06	1.50E-06
20	0.05	1.50E-07	9.10E-08
40	0.025	9.50E-09	5.90E-09

46

Supplemental Charts

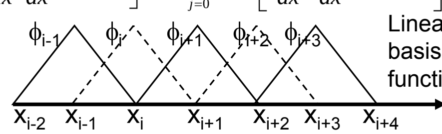
- Charts 48 to 52 show use of global approach to one-dimensional problem with linear shape functions
  - Gives same results as element analysis
    - Demonstrates need for assembly process
    - Not readily accomplished in 2D and 3D
- Charts 53 to 63 show details of analysis with quadratic shape functions
  - Same overall approach to getting $\Sigma A_{ij}u_j = 0$, but different equations for ϕ_i

47

Global Approach

- Get basis function for each i

$$\int_0^L \left[\frac{d\hat{u}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{u}\right]dx = \sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = 0$$



Linear basis functions

- Evaluate integral for each shape function, ϕ_i, to get an equation for each unknown, u_j

48

Global Interals

- Value of ϕ_i is nonzero only for $x_{i-1} \le x \le x_{i+1}$
- u_j coefficient zero unless j = i–1, i, or i+1

$$\sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx = A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} = 0$$

$$A_{ii-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1} \right] dx$$

$$A_{ii} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i \right] dx$$

$$A_{ii+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i+1} \right] dx$$

California State University Northridge 49

Results of Integration

$$\alpha_i = \frac{a^2}{3}(x_{i+1} - x_i) - \frac{1}{x_{i+1} - x_i} \qquad \beta_i = \frac{a^2}{6}(x_{i+1} - x_i) + \frac{1}{x_{i+1} - x_i}$$

$$\sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx = A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} = 0$$

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1} \right] dx = \beta_{i-1}$$

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i \right] dx = \alpha_i + \alpha_{i-1}$$

$$A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i+1} \right] dx = \beta_i$$

California State University Northridge 50

Evaluating the Integrals

- Use A_{ii-1} as an example

$$A_{ii-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1} \right] dx$$

$$A_{ii-1} = \int_{x_{i-1}}^{x_i} \left[\frac{-1}{x_i - x_{i-1}}\frac{1}{x_i - x_{i-1}} - \frac{x - x_{i-1}}{x_i - x_{i-1}} a^2 \frac{x_i - x}{x_i - x_{i-1}} \right] dx$$

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \end{cases} \qquad \varphi_{i-1}(x) = \begin{cases} \dfrac{x - x_{i-2}}{x_{i-1} - x_{i-2}} & x_{i-2} \le x \le x_{i-1} \\ \dfrac{x_i - x}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \end{cases}$$

California State University Northridge 51

Constant Steps $x_{i+1} – x_i = h$

$$\alpha_i = \frac{ha^2}{3} - \frac{1}{h} \qquad \beta_i = \frac{ha^2}{6} + \frac{1}{h}$$

$$\sum_{j=0}^{N} u_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j \right] dx = A_{i,i-1}u_{i-1} + A_{i,i}u_i + A_{i,i+1}u_{i+1} = 0$$

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1} \right] dx = \frac{ha^2}{6} + \frac{1}{h}$$

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i \right] dx = \frac{2ha^2}{3} - \frac{2}{h}$$

$$A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i+1} \right] dx = \frac{ha^2}{6} + \frac{1}{h}$$

California State University Northridge 52

Quadratic Shape Functions

- Use one-dimensional shape functions here as an example
- Need three nodes per element for quadratic shape function
  - Label nodes as i – 1, i, and i + 1
  - Define coordinates as shown below

$$h_i = x_i - x_{i-1} = x_{i+1} - x_i = \frac{x_{i+1} - x_{i-1}}{2}$$

$$\xi = 2\frac{x - x_i}{x_{i+1} - x_{i-1}} \qquad \xi = \frac{x - x_i}{h_i}$$

California State University Northridge 53

Quadratic Basis Functions II

$$\varphi_{i+1}(\xi) = \frac{\xi(\xi+1)}{2} = \frac{1}{2}\frac{x - x_i}{h_i}\left(\frac{x - x_i}{h_i} + 1\right)$$

$$\varphi_i(\xi) = 1 - \xi^2 = 1 - \left(\frac{x - x_i}{h_i}\right)^2$$

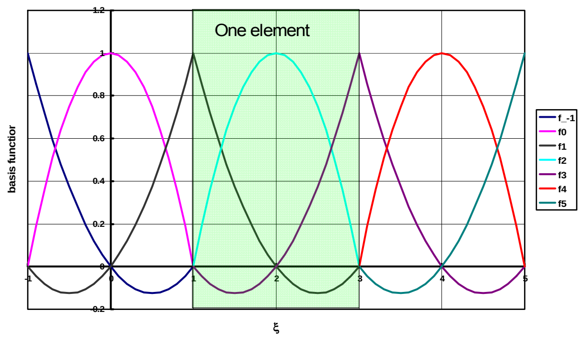
$$\varphi_{i-1}(\xi) = \frac{\xi(\xi-1)}{2} = \frac{1}{2}\frac{x - x_i}{h_i}\left(\frac{x - x_i}{h_i} - 1\right)$$

$$h_i = x_i - x_{i-1} = x_{i+1} - x_i = \frac{x_{i+1} - x_{i-1}}{2}$$

- Can use either definition of h_i in integrals
- Pick one to give simplest results

California State University Northridge 54

Quadratic Basis Functions, Global Basis



California State University Northridge 55

Quadratics for $d^2u/dx^2+a^2u = 0$

- Have three u_j coefficients in each element

$$\sum_{j=I}^{III} u_j \int_{x_I}^{x_{III}} \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = 0 \quad i = I, II, III$$

- Have three shape functions in each element
- Nine integrals to evaluate (3 u_j coefficients times 3 basis functions)
  - Three found by symmetry

California State University Northridge 56

Quadratic Element Results

- Result of element integrals

$$\sum_{j=I}^{III} u_j \int_{x_I}^{x_{III}} \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = 0 \quad i = I, II, III$$

$$\begin{bmatrix} \frac{7}{6h_i} - \frac{4a^2h_i}{15} & -\frac{4}{3h_i} - \frac{2a^2h_i}{15} & \frac{1}{6h_i} + \frac{a^2h_i}{15} \\ -\frac{4}{3h_i} - \frac{2a^2h_i}{15} & \frac{8}{3h_i} - \frac{16a^2h_i}{15} & -\frac{4}{3h_i} - \frac{2a^2h_i}{15} \\ \frac{1}{6h_i} + \frac{a^2h_i}{15} & -\frac{4}{3h_i} - \frac{2a^2h_i}{15} & \frac{7}{6h_i} - \frac{4a^2h_i}{15} \end{bmatrix} \begin{bmatrix} u_I \\ u_{II} \\ u_{III} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

California State University Northridge 57

Quadratic Assembly

- Nodes from i = 0 to i = N
- Elements from k = 0 to K = 2N – 2
- Element k has nodes 2k, 2k+1 and 2k+2
- Different equations for odd and even i

$$\left[-\frac{4}{3h_k} - \frac{2a^2h_k}{15}\right]u_{i-1} + \left[\frac{8}{3h_k} - \frac{16a^2h_k}{15}\right]u_i + \left[-\frac{4}{3h_k} - \frac{2a^2h_k}{15}\right]u_{i+1} = 0 \quad Odd\ i, k = \frac{i-1}{2}$$

California State University Northridge 58

Quadratic Assembly II

- Even i equation

$$\left[\frac{1}{6h_{k-1}} + \frac{a^2h_{k-1}}{15}\right]u_{i-2} + \left[-\frac{4}{3h_{k-1}} - \frac{2a^2h_{k-1}}{15}\right]u_{i-1} + \left[\frac{7}{6h_{k-1}} - \frac{4a^2h_{k-1}}{15} + \frac{7}{6h_k} - \frac{4a^2h_k}{15}\right]u_i + \left[-\frac{4}{3h_k} - \frac{2a^2h_k}{15}\right]u_{i+1} + \left[\frac{1}{6h_k} + \frac{a^2h_k}{15}\right]u_{i+2} = 0 \quad Even\ i \quad k = \frac{i}{2}$$

- Boundary nodes on next chart

California State University Northridge 59

Quadratic Assembly III

- Boundary nodes at i = 0 and i = N

$$\left[\frac{7}{6h_0} - \frac{4a^2h_0}{15}\right]u_0 + \left[-\frac{4}{3h_0} - \frac{2a^2h_0}{15}\right]u_1 + \left[\frac{1}{6h_0} + \frac{a^2h_0}{15}\right]u_2 = -\left.\frac{du}{dx}\right|_{x=0}$$

$$\left[\frac{7}{6h_K} - \frac{4a^2h_K}{15}\right]u_N + \left[-\frac{4}{3h_K} - \frac{2a^2h_K}{15}\right]u_{N-1} + \left[\frac{1}{6h_K} + \frac{a^2h_K}{15}\right]u_{N-2} = \left.\frac{du}{dx}\right|_{x=L}$$

- Can simplify equations for constant element size, h_k

California State University Northridge 60

Constant h

- Multiply all results by 3h or 6h to get whole number in lead term and a^2h^2 in remaining terms
  - Even i result below with odd i, i = 0, and i = N results on next chart

$$\left[1+\frac{6a^2h^2}{15}\right]u_{i-2}+\left[-8-\frac{12a^2h^2}{15}\right]u_{i-1}+\left[14-\frac{48a^2h^2}{15}\right]u_i+$$
$$\left[-8-\frac{12a^2h^2}{15}\right]u_{i+1}+\left[1+\frac{6a^2h^2}{15}\right]u_{i+2}=0 \quad Even\ \ i$$

California State University Northridge 61

Constant h (i = 0, N and odd i)

$$\left[7-\frac{24a^2h^2}{15}\right]u_0+\left[-8-\frac{12a^2h^2}{15}\right]u_1+\left[1+\frac{6a^2h^2}{15}\right]u_2=-h\left.\frac{du}{dx}\right|_{x=0}$$

$$\left[-4-\frac{6a^2h^2}{15}\right]u_{i-1}+\left[8-\frac{48a^2h^2}{15}\right]u_i$$
$$+\left[-4-\frac{6a^2h^2}{15}\right]u_{i+1}=0 \quad Odd\ \ i$$

$$\left[7-\frac{4a^2h^2}{15}\right]u_N+\left[-8-\frac{2a^2h^2}{15}\right]u_{N-1}+\left[1+\frac{a^2h^2}{15}\right]u_{N-2}=h\left.\frac{du}{dx}\right|_{x=L}$$

California State University Northridge 62

Quadratic Solutions

- Have N + 1 equations in N + 3 unknowns
- Add two boundary conditions equations
- Set of equations forms pentadiagonal matrix
- Solve by extension of Thomas algorithm that applies Gauss elimination to matrix with five diagonals next to each other

California State University Northridge 63