Additional Topics in Numerical Solutions of Elliptic Equations

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Mechanical Engineering 501B

Seminar in Engineering Analysis

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California State University Northridge

Outline

- Review last class
- Treatment of boundary conditions
  - Second kind (Neumann)
  - Third kind a∂u/∂s + bu = c
    - s is coordinate normal to boundary (x or y)
- Treatment using different gradient expressions
- Compact difference expressions for higher order accuracy

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Review Finite Differences

- Second-order second derivatives

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} = \frac{u_{i+1\,j} + u_{i-1\,j} - 2u_{ij}}{(\Delta x)^2} + O[(\Delta x)^2] \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{ij} = \frac{u_{ij+1} + u_{ij-1} - 2u_{ij}}{(\Delta y)^2} + \varepsilon$$

- Poisson-Laplace equation O[(Δx)², (Δy)²]

$$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\dot{Q}}{k}\right]_{ij} \approx \frac{u_{i+1\,j} + u_{i-1\,j} - 2u_{ij}}{(\Delta x)^2} + \frac{u_{ij+1} + u_{ij-1} - 2u_{ij}}{(\Delta y)^2} + \left(\frac{\dot{Q}}{k}\right)_{ij} = 0$$

- Multiply by (Δx)² and define β = Δx/Δy

$$u_{i+1\,j} + u_{i-1\,j} + \beta^2\left(u_{ij+1} + u_{ij-1}\right) - 2\left(1+\beta^2\right)u_{ij} + (\Delta x)^2\left(\frac{\dot{Q}}{k}\right)_{ij} = 0$$

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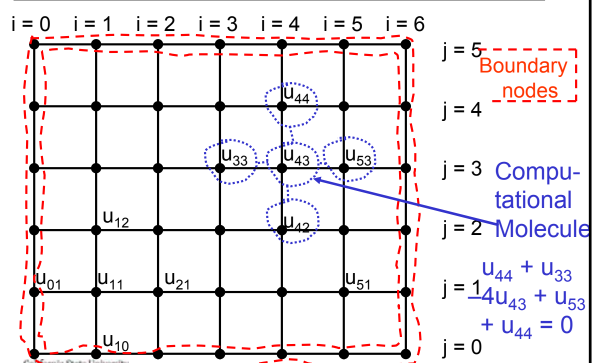
Review Small Grid (N = 6, M = 5)



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Review N = 6, M = 5 Matrix

$$\begin{bmatrix}
-4&1&&&&1&&&&&&&&&&&&&& \\
1&-4&1&&&&1&&&&&&&&&&&&& \\
&1&-4&1&&&&1&&&&&&&&&&&& \\
&&1&-4&1&&&&1&&&&&&&&&&& \\
&&&1&-4&&&&&1&&&&&&&&&& \\
1&&&&&-4&1&&&&1&&&&&&&&& \\
&1&&&&1&-4&1&&&&1&&&&&&&& \\
&&1&&&&1&-4&1&&&&1&&&&&&& \\
&&&1&&&&1&-4&1&&&&1&&&&&& \\
&&&&1&&&&1&-4&&&&&1&&&&& \\
&&&&&1&&&&&-4&1&&&&1&&&& \\
&&&&&&1&&&&1&-4&1&&&&1&&& \\
&&&&&&&1&&&&1&-4&1&&&&1&& \\
&&&&&&&&1&&&&1&-4&1&&&&1& \\
&&&&&&&&&1&&&&1&-4&&&&&1 \\
&&&&&&&&&&1&&&&&-4&1&&& \\
&&&&&&&&&&&1&&&&1&-4&1&& \\
&&&&&&&&&&&&1&&&&1&-4&1& \\
&&&&&&&&&&&&&1&&&&1&-4&1 \\
&&&&&&&&&&&&&&1&&&&1&-4
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{51} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \\ u_{52} \\ u_{13} \\ u_{23} \\ u_{33} \\ u_{43} \\ u_{53} \\ u_{14} \\ u_{24} \\ u_{34} \\ u_{44} \\ u_{54} \end{bmatrix}$$

Zero coefficients because of x boundary

$u_{44} + u_{33} + -4u_{43} + u_{53} + u_{44} = 0$

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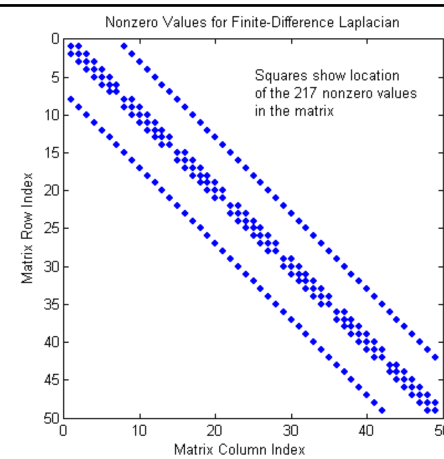

Review Iterative Solutions

- Jacobi iteration uses all old values

$$u_{ij}^{(n+1)} = b'_{ij} - A_{ij}^{S'} u_{ij-1}^{(n)} - A_{ij}^{W'} u_{i-1j}^{(n)} - A_{ij}^{E'} u_{i+1j}^{(n)} - A_{ij}^{N'} u_{ij+1}^{(n)}$$

- Gauss–Seidel uses most-recent values

$$u_{ij}^{(n+1)} = b'_{ij} - A_{ij}^{S'} u_{ij-1}^{(n+1)} - A_{ij}^{W'} u_{i-1j}^{(n+1)} - A_{ij}^{E'} u_{i+1j}^{(n)} - A_{ij}^{N'} u_{ij+1}^{(n)}$$

- Relaxation

$$u_{ij}^{(n+1)} = (1-\omega) u_{ij}^{(n)} - \omega \Big[A_{ij}^{S'} u_{ij-1}^{(n+1)} + A_{ij}^{W'} u_{i-1j}^{(n+1)} + A_{ij}^{E'} u_{i+1j}^{(n)} + A_{ij}^{N'} u_{ij+1}^{(n)} - b'_{ij} \Big]$$

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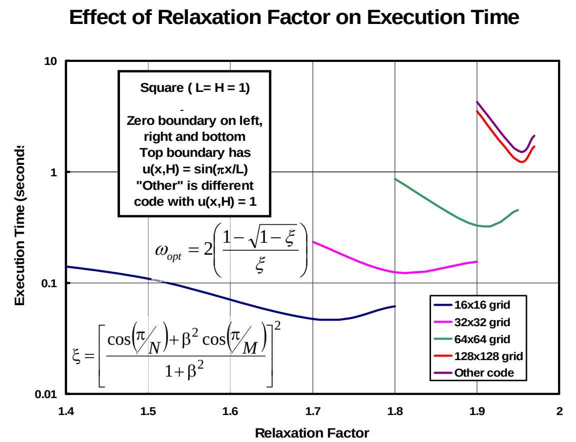

Review Errors

$$(\text{Relative Change})_{ij}^{(n+1)} = \left| \frac{u_{ij}^{(n+1)} - u_{ij}^{(n)}}{u_{ij}^{(n+1)}} \right|$$

Computable

$$(\text{Residual})_{ij}^{(n+1)} = -u_{ij}^{(n+1)} + A_{ij}^{S'} u_{ij-1}^{(n+1)} + A_{ij}^{W'} u_{i-1j}^{(n+1)} + A_{ij}^{E'} u_{i+1j}^{(n+1)} + A_{ij}^{N'} u_{ij+1}^{(n+1)} - b'_{ij}$$

$$(\text{Iteration Error})_{ij}^{(n+1)} = u_{ij}^{(\infty)} - u_{ij}^{(n+1)}$$

$$(\text{Exact Error})_{ij}^{(n+1)} = u_{PDE}(x_i, y_j) - u_{ij}^{(n+1)}$$

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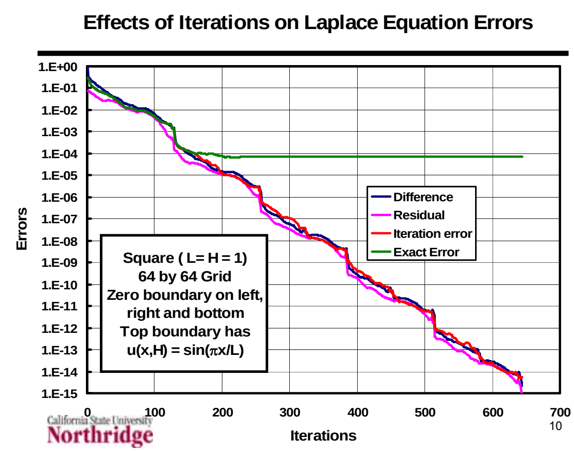

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Other Boundary Conditions

- General condition a∂u/∂s + bu = c
  - a = 0, b = 1 for Dirichlet (value given)
  - a = 1, b = 0 for Neumann (gradient given)
  - Mixed has both a and b nonzero
- Write general boundary condition using a finite difference expression for ∂u/∂s
  - Two approaches
    - Using second order forward or backward difference for boundary node
    - Add fictitious node outside boundary and use central differences at boundary

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Notation

- Label the boundaries at x = x_0 and x = x_N as the (W)est and (E)ast boundaries
  - a, b, and c can be different at each y_j
  - Notation: a_j^W, b_j^W, c_j^W, a_j^E, b_j^E, and c_j^E
- Boundaries at y = y_0 and y = y_M are the (S)outh and (N)orth boundaries
  - a, b, and c can be different at each x_i
  - Notation: a_i^S, b_i^S, c_i^S, a_i^N, b_i^N, and c_i^N

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Central Difference Approach

- General condition $a\partial u/\partial s + bu = c$ by central differences
  - Write general equation for boundary nodes that contains fictitious node outside region
  - Write central difference equation for boundary condition
  - Solve this equation for potential at fictitious node and use result to replace this value in general equation at boundary
  - Include boundary nodes in iterations

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(N)orth Boundary Example

- Finite-difference equation at $y = y_M$

$$A^S_{iM}u_{iM-1} + A^W_{iM}u_{i-1M} + A^P_{iM}u_{iM} + A^E_{iM}u_{i+1M} + A^N_{iM}u_{iM+1} = Q_{iM}$$

- Fictitious u_{iM+1} from central-difference boundary condition equation at $y = y_M$

$$a_i^N \frac{u_{iM+1} - u_{iM-1}}{2\Delta y} + b_i^N u_{iM} = c_i^N$$

$$u_{iM+1} = u_{iM-1} + \frac{2\Delta y}{a_i^N}\left(c_i^N - b_i^N u_{iM}\right)$$

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(N)orth Boundary Example II

- Combine equations to eliminate u_{iM+1}

$$A^S_{iM}u_{iM-1} + A^W_{iM}u_{i-1M} + A^P_{iM}u_{iM} + A^E_{iM}u_{i+1M} + A^N_{iM}\left[u_{iM-1} + \frac{2\Delta y}{a_i^N}\left(c_i^N - b_i^N u_{iM}\right)\right] = Q_{iM}$$

$$a_i^N\left(A^S_{iM} + A^N_{iM}\right)u_{iM-1} + a_i^N A^W_{iM}u_{i-1M} + \left(a_i^N A^P_{iM} - 2\Delta y A^N_{iM} b_i^N\right)u_{iM} + a_i^N A^E_{iM}u_{i+1M} = a_i^N Q_{iM} - 2\Delta y A^N_{iM} c_i^N$$

- Modified coefficients at (iterated) boundaries: new A^S, A^P, and Q; $A^N = 0$; A^W and A^E multiplied by a^N

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CD General Boundary

- Modified east boundary coefficients

$$A^P_{Nj} \leftarrow a_j^E A^P_{Nj} - A^E_{Nj} 2\Delta x b_j^E \qquad A^W_{Nj} \leftarrow a_j^E\left(A^E_{Nj} + A^W_{Nj}\right)$$

$$Q_{Nj} \leftarrow a_j^E Q_{Nj} - 2\Delta x c_j^E A^W_{Nj} \qquad A^N_{Nj} \leftarrow a_j^E A^N_{Nj}$$

$$A^S_{Nj} \leftarrow a_j^E A^S_{Nj} \qquad A^E_{Nj} \leftarrow 0$$

- Modified north boundary coefficients

$$A^P_{iM} \leftarrow a_i^N A^P_{iM} - A^N_{iM} 2\Delta y b_i^N \qquad A^S_{iM} \leftarrow a_i^N\left(A^N_{iM} + A^S_{iM}\right)$$

$$Q_{iM} \leftarrow a_i^N Q_{iM} - 2\Delta y c_i^N A^N_{iM} \qquad A^E_{iM} \leftarrow a_i^N A^E_{iM}$$

$$A^W_{iM} \leftarrow a_i^N A^W_{iM} \qquad A^N_{iM} \leftarrow 0$$

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CD General Boundary II

- Modified west boundary coefficients

$$A^P_{0j} \leftarrow a_j^W A^P_{0j} + A^W_{0j} 2\Delta x b_j^W \qquad A^E_{0j} \leftarrow a_j^W\left(A^E_{0j} + A^W_{0j}\right)$$

$$Q_{0j} \leftarrow a_j^W Q_{0j} + 2\Delta x c_j^W A^W_{0j} \qquad A^N_{0j} \leftarrow a_j^W A^N_{0j}$$

$$A^S_{0j} \leftarrow a_j^W A^S_{0j} \qquad A^W_{0j} \leftarrow 0$$

- Modified south boundary coefficients

$$A^P_{i0} \leftarrow a_i^S A^P_{i0} + A^S_{i0} 2\Delta y b_i^S \qquad A^N_{i0} \leftarrow a_i^S\left(A^N_{i0} + A^S_{i0}\right)$$

$$Q_{i0} \leftarrow a_i^S Q_{i0} + 2\Delta y c_i^S A^S_{i0} \qquad A^E_{i0} \leftarrow a_i^S A^E_{i0}$$

$$A^W_{i0} \leftarrow a_i^S A^W_{i0} \qquad A^S_{i0} \leftarrow 0$$

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Forward/Backward Difference

- General condition, $a\partial u/\partial s + bu = c$
  - Use forward differences at i = 0 or j = 0 and backward differences at i = N or j = M
  - Obtain equation for boundary potential in terms of two nodes in from boundary
  - Combine this equation with general equation for first node in from the boundary to eliminate unknown boundary potential
  - No iteration on boundary values, which are found at end of iterations
  - Derivation at end of this presentation

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(W)est and (S)outh Boundary

$$A_{1j}^P \leftarrow A_{1j}^P - A_{1j}^W \frac{4a_{Wj}}{2\Delta x b_{Wj} - 3a_{Wj}} \qquad A_{1j}^E \leftarrow A_{1j}^E + \frac{a_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W}$$

$$Q_{1j} \leftarrow Q_{1j} - \frac{2\Delta x c_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W} \qquad A_{1j}^W \leftarrow 0$$

$$A_{1j}^N \text{ and } A_{1j}^S \text{ unchanged} \qquad u_{0j} = \frac{2\Delta x c_j^W - 4a_j^W u_{1j} + a_j^W u_{2j}}{2\Delta x b_j^W - 3a_j^W}$$

$$A_{i1}^P \leftarrow A_{i1}^P - \frac{4a_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S} \qquad A_{i1}^N \leftarrow A_{i1}^N + \frac{a_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S}$$

$$Q_{i1} \leftarrow Q_{i1} - \frac{2\Delta y c_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S} \qquad A_{i1}^S \leftarrow 0$$

$$u_{i0} = \frac{2\Delta y c_i^S - 4a_i^S u_{i1} + a_i^S u_{i2}}{2\Delta y b_i^S - 3a_i^S}$$

$$A_{i1}^E \text{ and } A_{i1}^W \text{ unchanged}$$

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19

(N)orth and (E)ast Boundary

$$A_{iM-1}^P \leftarrow A_{iM-1}^P + \frac{4a_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N} \qquad A_{iM-1}^S \leftarrow A_{iM-1}^S - \frac{a_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N}$$

$$Q_{iM-1} \leftarrow Q_{iM-1} - \frac{2\Delta y c_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N} \qquad A_{iM-1}^N \leftarrow 0$$

$$u_{iM} = \frac{2\Delta y c_i^N + 4a_i^N u_{iM-1} - a_i^N u_{iM-2}}{2\Delta y b_i^N + 3a_i^N}$$

$$A_{iM-1}^E \text{ and } A_{iM-1}^W \text{ unchanged}$$

$$A_{N-1j}^W \leftarrow A_{N-1j}^W - \frac{a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E} \qquad A_{N-1j}^P \leftarrow A_{N-1j}^P + \frac{4a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}$$

$$Q_{N-1j} \leftarrow Q_{N-1j} - \frac{2\Delta x c_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E} \qquad A_{N-1j}^E \leftarrow 0$$

$$u_{Nj} \; \frac{2\Delta x c_j^E + 4a_j^E u_{N-1j} - a_j^E u_{N-2j}}{2\Delta x b_j^E + 3a_j^E}$$

$$A_{N-1j}^N \text{ and } A_{N-1j}^S \text{ unchanged}$$

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20

Compact Approximations

- Define central-difference operator, δ^2

$$\delta^2 f_i = f_{i+1} + f_{i-1} - 2 f_i$$

- Compact second derivative

$$f_i'' = \frac{\delta^2 f_i}{(\Delta x)^2 \left(1 + \frac{\delta^2}{12}\right)} + O\left[(\Delta x)^4\right]$$

- To apply this equation, we have to multiply through by denominator and then apply the operator there

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21

Final Result

$$u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1} + 20u_{ij} - \frac{10\beta^2 - 2}{1+\beta^2}\left(u_{i+1j} + u_{i-1j}\right)$$

$$\frac{10 - 2\beta^2}{1+\beta^2}\left(u_{ij+1} + u_{ij-1}\right) - \frac{(\Delta y)^2}{12\left(1+\beta^2\right)}\left[100Q_{ij} + 10\left(Q_{i+1j} + Q_{i-1j}\right.\right.$$

$$\left.\left. + Q_{ij+1} + Q_{ij-1}\right) + Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right] = 0$$

- Each value of u_{ij} is linked to eight nearest neighbors so each finite difference equation has nine terms
  - Uses weighted average source term
  - Derivation at end of this presentation

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Final Result for Δx = Δy = h

- For Δx = Δy, β = 1

$$u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1} - 20u_{ij} + \frac{10\beta^2 - 2}{1+\beta^2}\left(u_{i+1j} + u_{i-1j}\right)$$

$$\frac{10 - 2\beta^2}{1+\beta^2}\left(u_{ij+1} + u_{ij-1}\right) - \frac{(\Delta y)^2}{12\left(1-\beta^2\right)}\left[100Q_{ij} + 10\left(Q_{i+1j} + Q_{i-1j}\right.\right.$$

$$\left.\left. + Q_{ij+1} + Q_{ij-1}\right) + Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right] = 0$$

$$u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1} - 20u_{ij} + 4\left(u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1}\right)$$

$$-\frac{h^2}{24}\left[100Q_{ij} + 10\left(Q_{i+1j} + Q_{i-1j} + Q_{ij+1} + Q_{ij-1}\right)\right.$$

$$\left. + Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right] = 0$$

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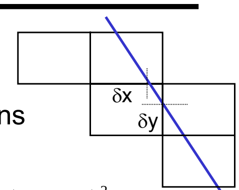
23

Boundary Crosses Grid

- Define new δx and δy to define boundary
- Get derivative expressions for uneven grid spacing



$$f_i' = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} - \frac{f_i''}{2!}\frac{(x_{i+1} - x_i)^2 - (x_{i-1} - x_i)^2}{x_{i+1} - x_{i-1}} + \ldots..$$

$$f_i'' = \frac{\dfrac{f_{i+1} - f_i}{x_{i+1} - x_i} - \dfrac{f_i - f_{i-1}}{x_i - x_{i-1}}}{\dfrac{x_{i+1} + x_i}{2} - \dfrac{x_i + x_{i-1}}{2}} + O(h)$$

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24

Boundary Crosses Grid II

- Simplify f'' equation as follows

$$f_i''=\frac{\frac{f_{i+1}-f_i}{x_{i+1}-x_i}-\frac{f_i-f_{i-1}}{x_i-x_{i-1}}}{\frac{x_{i+1}+x_i}{2}-\frac{x_i+x_{i-1}}{2}}=2\frac{\frac{f_{i+1}-f_i}{x_{i+1}-x_i}-\frac{f_i-f_{i-1}}{x_i-x_{i-1}}}{x_{i+1}+x_i-(x_i+x_{i-1})}$$

$$=(x_{i+1}-x_{i-1})$$

$$f_i''=\frac{2}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}f_{i+1}+\frac{2}{(x_i-x_{i-1})(x_{i+1}-x_{i-1})}f_{i-1}-\frac{2}{x_{i+1}-x_{i-1}}\left[\frac{1}{x_{i+1}-x_i}+\frac{1}{x_i-x_{i-1}}\right]f_i$$

- Use for Laplace equation ∂u/∂x, ∂u/∂y

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Boundary Crosses Grid III

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\approx\frac{2}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}u_{i+1j}+\frac{2}{(x_i-x_{i-1})(x_{i+1}-x_{i-1})}u_{i-1j}$$
$$+\frac{2}{(y_{j+1}-y_j)(y_{j+1}-y_{j-1})}u_{ij+1}+\frac{2}{(y_j-y_{j-1})(y_{j+1}-y_{j-1})}u_{ij-1}$$
$$-\left\{\frac{2}{y_{j+1}-y_{j-1}}\left[\frac{1}{y_{j+1}-y_j}+\frac{1}{y_j-y_{j-1}}\right]+\frac{2}{x_{i+1}-x_{i-1}}\left[\frac{1}{x_{i+1}-x_i}+\frac{1}{x_i-x_{i-1}}\right]\right\}u_{ij}=0$$

$$A_i^W=\frac{2}{(x_i-x_{i-1})(x_{i+1}-x_{i-1})}\qquad A_i^E=\frac{2}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})}$$
$$A_j^S=\frac{2}{(y_j-y_{j-1})(y_{j+1}-y_{j-1})}\qquad A_j^N=\frac{2}{(y_{j+1}-y_j)(y_{j+1}-y_{j-1})}$$
$$A_j^P=-A_j^N-A_j^S-A_i^E-A_i^W$$
$$A_j^S u_{ij-1}+A_i^W u_{i-1j}+A_j^P u_{ij}+A_i^E u_{i+1j}+A_j^N u_{ij+1}=0$$

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Boundary Crosses Grid IV

- Hoffman uses with different notation
  - Define $\Delta x_+ = x_{i+1} - x_i$ and $\Delta x_- = x_i - x_{i-1}$
  - Define $\Delta y_+ = y_{j+1} - y_j$ and $\Delta y_- = y_j - y_{j-1}$

$$A_{ij}^{+}u_{i+1j}-2A_{ij}^{0}u_{ij}+A_{ij}^{-}u_{i-1j}+B_{ij}^{+}u_{ij+1}-2B_{ij}^{0}u_{ij}+B_{ij}^{-}u_{ij-1}=0$$

$$A_{ij}^{+}=A_{ij}^{E}=\frac{2\Delta x_-}{\Delta x_-(\Delta x_+)^2+\Delta x_+(\Delta x_-)^2}\qquad A_{ij}^{-}=A_{ij}^{W}=\frac{2\Delta x_+}{\Delta x_-(\Delta x_+)^2+\Delta x_+(\Delta x_-)^2}$$
$$B_{ij}^{+}=A_{ij}^{N}=\frac{2\Delta y_-}{\Delta y_-(\Delta y_+)^2+\Delta y_+(\Delta y_-)^2}\qquad B_{ij}^{-}=A_{ij}^{S}=\frac{2\Delta y_+}{\Delta y_-(\Delta y_+)^2+\Delta y_+(\Delta y_-)^2}$$
$$A_{ij}^{P}=-2A_{ij}^{-}-2B_{ij}^{-}=\frac{-2(\Delta x_-+\Delta x_+)}{\Delta x_-(\Delta x_+)^2+\Delta x_+(\Delta x_-)^2}+\frac{-2(\Delta y_-+\Delta y_+)}{\Delta y_-(\Delta y_+)^2+\Delta y_+(\Delta y_-)^2}$$

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Boundary Crosses Grid V

- Use uneven finite-difference expressions in differential equations
- Can create problems with stability in explicit procedures
- Care is required in modeling gradient boundary conditions
- Generally not favored
  - Exception is Flow-3D software by C. W. "Tony" Hirt who recommends this procedure

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Summary

- Elliptic finite-difference formed in the same way as parabolic ones
- Equations require iterative solvers
- Simple solvers will work well for small problems
- Advanced solvers required for more complex problems
- Need to treat boundaries that do not fall on grid nodes

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Midterm

- Next Wednesday, April 1
- Covers wave equation, PDEs in more than two independent variables, classification of PDEs and numerical analysis introduction
- Problems similar to previous midterm and homework
- Any questions?

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Additional Materials

- Slides 32 to 40 present detailed derivations of the general boundary conditions using one-sided differences
  - See Hoffman for other derivations of central-difference boundary equations
- The derivation of the higher order difference method using compact differences is shown on slides 41 to 47

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General Boundary Example

- Look at (W)est ($x = x_0$) boundary; other boundaries follow similar derivation
- Solve one-sided, second-order first derivative expression for u_{0j}

$$a_j^W \left.\frac{\partial u}{\partial x}\right|_{x=x_0} + b_j^W u_{x=x_0} = a_j^W \frac{-3u_{0j} + 4u_{1j} - u_{2j}}{2\Delta x} + b_j^W u_{0j} = c_j^W$$

$$\left(2\Delta x b_j^W - 3a_j^W\right)u_{0j} + 4a_j^W u_{1j} - a_j^W u_{2j} = 2\Delta x c_j^W$$

$$u_{0j} = \frac{2\Delta x c_j^W - 4a_j^W u_{1j} + a_j^W u_{2j}}{2\Delta x b_j^W - 3a_j^W}$$

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General Boundary Example II

- Combine boundary condition with equation for first node from boundary

$$A_{1j}^S u_{1j-1} + A_{1j}^W u_{0j} + A_{1j}^P u_{1j} + A_{1j}^E u_{2j} + A_{1j}^N u_{1j+1} = Q_{1j}$$

$$A_{1j}^S u_{1j-1} + A_{1j}^W \frac{2\Delta x c_j^W - 4a_j^W u_{1j} + a_j^W u_{2j}}{2\Delta x b_j^W - 3a_j^W} + A_{1j}^P u_{1j} + A_{1j}^E u_{2j} + A_{1j}^N u_{1j+1} = Q_{1j}$$

$$A_{1j}^S u_{1j-1} + \left(A_{1j}^P - A_{1j}^W \frac{4a_{Wj}}{2\Delta x b_{Wj} - 3a_{Wj}}\right) u_{1j}$$

$$+ \left(A_{1j}^E + \frac{a_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W}\right) u_{2j} + A_{1j}^N u_{1j+1} = Q_{1j} - \frac{2\Delta x c_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W}$$

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General Boundary Example III

- Boundary node no longer in equation
  - Still solve for interior nodes only with solver
  - Find boundary value after solution complete
  - Use modified equation coefficients

$$A_{1j}^P \leftarrow A_{1j}^P - A_{1j}^W \frac{4a_{Wj}}{2\Delta x b_{Wj} - 3a_{Wj}} \qquad A_{1j}^E \leftarrow A_{1j}^E + \frac{a_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W}$$

$$Q_{1j} \leftarrow Q_{1j} - \frac{2\Delta x c_j^W A_{1j}^W}{2\Delta x b_j^W - 3a_j^W} \qquad A_{1j}^W \leftarrow 0$$

$$u_{0j} = \frac{2\Delta x c_j^W - 4a_j^W u_{1j} + a_j^W u_{2j}}{2\Delta x b_j^W - 3a_j^W}$$

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Results for Other Boundaries

- Substitute revised coefficients for original coefficients in finite-difference version of PDE
- Similar approach, with different equations, at all boundaries
- Iterate only on nonboundary nodes and find boundary potentials after iterations complete
- Derivations follow here

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South ($y = y_0$) Boundary

- Result similar to (W)est ($x = x_0$) boundary

$$A_{i1}^S u_{i0} + A_{i1}^W u_{i-11} + A_{i1}^P u_{i1} + A_{i1}^E u_{i+11} + A_{i1}^N u_{i2} = Q_{i1}$$

$$A_{i1}^S \frac{2\Delta y c_i^S - 4a_i^S u_{i1} + a_i^S u_{i2}}{2\Delta y b_i^S - 3a_i^S} + A_{i1}^W u_{i-11} + A_{i1}^P u_{i1} + A_{i1}^E u_{i2} + A_{i1}^N u_{i2} = Q_{i1}$$

$$A_{i1}^W u_{i-11} + \left(A_{i1}^P - \frac{4a_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S}\right) u_{i1} + A_{i1}^E u_{i+11}$$

$$+ \left(A_{i+11}^N + \frac{a_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S} + \right) u_{i2} = Q_{i1} - \frac{2\Delta y c_i^S A_{i1}^S}{2\Delta y b_i^S - 3a_i^S}$$

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(N)orth and (E)ast Boundaries

- Same approach with backward difference expressions
  - Look at (E)ast as example

$$a_j^E \left.\frac{\partial y}{\partial x}\right|_{x=x_N} + b_j^E u_{Nj} = a_j^E \frac{3u_{Nj} - 4u_{N-1j} + u_{N-2j}}{2\Delta x} + b_j^E u_{Nj} = c_j^E$$

$$\left(2\Delta x b_j^E + 3a_j^E\right)u_{Nj} - 4a_j^E u_{N-1j} + a_j^E u_{N-2j} = 2\Delta x c_j^E$$

$$u_{Nj} = \frac{2\Delta x c_j^E + 4a_j^E u_{N-1j} - a_j^E u_{N-2j}}{2\Delta x b_j^E + 3a_j^E}$$

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(E)ast Boundary

- Combine boundary condition with equation for last node before boundary

$$A_{N-1j}^S u_{N-1j-1} + A_{N-1j}^W u_{N-2j} + A_{N-1j}^P u_{N-1j} + A_{N-1j}^E u_{Nj} + A_{N-1j}^N u_{N-1j+1} = Q_{N-1j}$$

$$A_{N-1j}^S u_{N-1j-1} + A_{N-1j}^W u_{N-2j} + A_{N-1j}^P u_{N-1j} +$$
$$A_{N-1j}^E \frac{2\Delta x c_j^E + 4a_j^E u_{N-1j} - a_j^E u_{N-2j}}{2\Delta x b_j^E + 3a_j^E} + A_{N-1j}^N u_{N-1j+1} = Q_{N-1j}$$

$$A_{N-1j}^S u_{N-1j-1} + \left(A_{N-1j}^W - \frac{a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}\right)u_{N-2j} + \left(A_{N-1j}^P + \frac{4a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}\right)u_{N-1j}$$
$$+ A_{N-1j}^N u_{N-1j+1} = Q_{N-1j} - \frac{2\Delta x c_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}$$

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(E)ast Boundary II

- Final results

$$A_{N-1j}^S u_{N-1j-1} + \left(A_{N-1j}^W - \frac{a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}\right)u_{N-2j} + \left(A_{N-1j}^P + \frac{4a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}\right)u_{N-1j}$$
$$+ A_{N-1j}^N u_{N-1j+1} = Q_{N-1j} - \frac{2\Delta x c_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}$$

$$A_{N-1j}^W \leftarrow A_{N-1j}^W - \frac{a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E} \qquad A_{N-1j}^P \leftarrow A_{N-1j}^P + \frac{4a_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E}$$

$$Q_{N-1j} \leftarrow Q_{N-1j} - \frac{2\Delta x c_j^E A_{N-1j}^E}{2\Delta x b_j^E + 3a_j^E} \qquad A_{N-1j}^E \leftarrow 0$$

$$u_{Nj} \frac{2\Delta x c_j^E + 4a_j^E u_{N-1j} - a_j^E u_{N-2j}}{2\Delta x b_j^E + 3a_j^E}$$

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(N)orth Boundary

- Result for y = y_M boundary similar to x = x_N

$$A_{iM-1}^S u_{iM-2} + A_{iM-1}^W u_{i-1M-1} + A_{iM-1}^P u_{iM-1} + A_{iM-1}^E u_{i+1M-1} + A_{iM-1}^N u_{iM} = Q_{iM-1}$$

$$A_{iM-1}^S u_{iM-2} + A_{iM-1}^W u_{i-1M-1} + A_{iM-1}^P u_{iM-1} + A_{iM-1}^E u_{i+1M-1}$$
$$+ A_{iM-1}^N \frac{2\Delta y c_i^N + 4a_i^N u_{iM-1} - a_i^N u_{iM-2}}{2\Delta y b_i^N + 3a_i^N} = Q_{iM-1}$$

$$\left(A_{iM-1}^S - \frac{a_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N}\right)u_{iM-2} + A_{iM-1}^W u_{i-1M-1} +$$
$$\left(A_{iM-1}^P + \frac{4a_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N}\right)u_{iM-1} + A_{iM-1}^E u_{i+1M-1} = Q_{iM-1} - \frac{2\Delta y c_i^N A_{iM-1}^N}{2\Delta y b_i^N + 3a_i^N}$$

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PDE Compact Differences

- Compact differences for Poisson equation
  - Define x and y direction difference operators
  - Second partial derivative expressions

$$\delta_x^2 g_{ij} = g_{i+1j} + g_{i-1j} - 2g_{ij} \qquad \delta_y^2 g_{ij} = g_{ij+1} + g_{ij-1} - 2g_{ij}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} = \frac{\delta_x^2 u_i}{(\Delta x)^2\left(1 + \frac{\delta_x^2}{12}\right)} + O\left[(\Delta x)^4\right] \qquad \left.\frac{\partial^2 u}{\partial y^2}\right|_{ij} = \frac{\delta_y^2 u_i}{(\Delta y)^2\left(1 + \frac{\delta_y^2}{12}\right)} + O\left[(\Delta y)^4\right]$$

- Apply to $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = Q(x,y)$

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Poisson Compact Differences

$$0 = \left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{ij} - Q_{ij} = \frac{\delta_x^2 u_{ij}}{(\Delta x)^2\left(1 + \frac{\delta_x^2}{12}\right)} + \frac{\delta_y^2 u_{ij}}{(\Delta y)^2\left(1 + \frac{\delta_y^2}{12}\right)} - Q_{ij}$$

$$= (\Delta y)^2\left(1 + \frac{\delta_y^2}{12}\right)\delta_x^2 u_{ij} + (\Delta x)^2\left(1 + \frac{\delta_x^2}{12}\right)\delta_y^2 u_{ij} - (\Delta y)^2\left(1 + \frac{\delta_y^2}{12}\right)(\Delta x)^2\left(1 + \frac{\delta_x^2}{12}\right)Q_{ij}$$

- Multiply by $12/(\Delta x)^2$ and use $\beta = \Delta y/\Delta x$

$$\beta^2\left(12 + \delta_y^2\right)\delta_x^2 u_{ij} + \left(1 + \delta_x^2\right)\delta_y^2 u_{ij} - (\Delta y)^2\left(12 + \delta_y^2\right)\left(1 + \frac{\delta_x^2}{12}\right)Q_{ij} = 0$$

- We know $\delta_x^2 u_{ij}$ and $\delta_y^2 u_{ij}$, but what is $\delta_x^2 \delta_y^2 u_{ij}$

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Mixed Difference Operators

- Apply each operator in order
  - Get same result regardless of order

$$\delta_x^2\delta_y^2 u_{ij} = \delta_x^2\left(\delta_y^2 u_{ij}\right) = \delta_x^2\left(u_{ij+1} + u_{ij-1} - 2u_{ij}\right)$$

$$\delta_x^2\delta_y^2 u_{ij} = u_{i+1j+1} + u_{i-1j+1} - 2u_{ij+1} + u_{i+1j-1} + u_{i-1j-1} - 2u_{ij-1} - 2\left(u_{i+1j} + u_{i-1j} - 2u_{ij}\right)$$

$$\delta_y^2\delta_x^2 u_{ij} = \delta_y^2\left(\delta_x^2 u_{ij}\right) = \delta_y^2\left(u_{i+1j} + u_{i-1j} - 2u_{ij}\right)$$

$$\delta_y^2\delta_x^2 u_{ij} = u_{i+1j+1} + u_{i+1j-1} - 2u_{i+1j} + u_{i-1j+1} + u_{i-1j-1} - 2u_{i-1j} - 2\left(u_{ij+1} + u_{ij-1} - 2u_{ij}\right)$$

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Back to Original Equation

- Reduce basic finite-difference equation

$$\beta^2\left(12+\delta_y^2\right)\delta_x^2 u_{ij} + \left(12+\delta_x^2\right)\delta_y^2 u_{ij} - (\Delta y)^2\left(12+\delta_y^2\right)\left(1+\frac{\delta_x^2}{12}\right)Q_{ij} = 0$$

$$\beta^2\left(12+\delta_y^2\right)\delta_x^2 u_{ij} + \left(12+\delta_x^2\right)\delta_y^2 u_{ij} = 12\left(\beta^2\delta_x^2 u_{ij} + \delta_y^2 u_{ij}\right) + \left(1+\beta^2\right)\delta_y^2\delta_x^2 u_{ij}$$

$$= 12\beta^2\left(u_{i+1j} + u_{i-1j} - 2u_{ij}\right) + 12\left(u_{ij+1} + u_{ij-1} - 2u_{ij}\right) + \left(1+\beta^2\right)\left(u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1} - 2u_{i-1j} - 2u_{i+1j} - 2u_{ij+1} - 2u_{ij-1} + 4u_{ij}\right)$$

$$= \left(1+\beta^2\right)\left(u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1}\right) - 20\left(1+\beta^2\right)u_{ij} + \left(10\beta^2 - 2\right)\left(u_{i+1j} + u_{i-1j}\right) + \left(10 - 2\beta^2\right)\left(u_{ij+1} + u_{ij-1}\right)$$

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Original Equation Source Term

- Continue reduction of basic equation

$$\beta^2\left(12+\delta_y^2\right)\delta_x^2 u_i + \left(12+\delta_x^2\right)\delta_y^2 u_i - (\Delta y)^2\left(12+\delta_y^2\right)\left(1+\frac{\delta_x^2}{12}\right)Q_{ij} = 0$$

$$\left(12+\delta_y^2\right)\left(1+\frac{\delta_x^2}{12}\right)Q_{ij} = 12Q_{ij} + \delta_x^2 Q_{ij} + \delta_y^2 Q_{ij} + \frac{1}{12}\delta_y^2\delta_x^2 Q_{ij}$$

$$= 12Q_{ij} + \left(Q_{i+1j} + Q_{i-1j} - 2Q_{ij}\right) + \left(Q_{ij+1} + Q_{ij-1} - 2Q_{ij}\right) + \frac{1}{12}\left(Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1} - 2Q_{i-1j} - 2Q_{i+1j} - 2Q_{ij+1} - 2Q_{ij-1} + 4Q_{ij}\right)$$

$$= \frac{25}{3}Q_{ij} + \frac{5}{6}\left(Q_{i+1j} + Q_{i-1j} + Q_{ij+1} + Q_{ij-1}\right)$$

$$\frac{1}{12}\left(Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right)$$

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Combine Results

$$\beta^2\left(12+\delta_y^2\right)\delta_x^2 u_i + \left(12+\delta_x^2\right)\delta_y^2 u_i - (\Delta y)^2\left(12+\delta_y^2\right)\left(1+\frac{\delta_x^2}{12}\right)Q_{ij} = 0$$

$$\left(1+\beta^2\right)\left(u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1}\right) - 20\left(1+\beta^2\right)u_{ij} + \left(10\beta^2 - 2\right)\left(u_{i+1j} + u_{i-1j}\right) + \left(10 - 2\beta^2\right)\left(u_{ij+1} + u_{ij-1}\right)$$
$$- (\Delta y)^2\left[\frac{25}{3}Q_{ij} + \frac{5}{6}\left(Q_{i+1j} + Q_{i-1j} + Q_{ij+1} + Q_{ij-1}\right)\right] - \frac{(\Delta y)^2}{12}\left(Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right)\Bigg] = 0$$

- Divide by 1 + β^2 and rearrange

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Combine Results II

$$\left(u_{i+1j+1} + u_{i+1j-1} + u_{i-1j+1} + u_{i-1j-1}\right) - 20u_{ij} + \frac{10\beta^2 - 2}{1+\beta^2}\left(u_{i+1j} + u_{i-1j}\right) + \frac{10 - 2\beta^2}{1+\beta^2}\left(u_{ij+1} + u_{ij-1}\right)$$
$$- \frac{(\Delta y)^2}{1+\beta^2}\left[\frac{25}{3}Q_{ij} + \frac{5}{6}\left(Q_{i+1j} + Q_{i-1j} + Q_{ij+1} + Q_{ij-1}\right)\right] - \frac{(\Delta y)^2}{12\left(1+\beta^2\right)}\left(Q_{i+1j+1} + Q_{i+1j-1} + Q_{i-1j+1} + Q_{i-1j-1}\right)\Bigg] = 0$$

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