Midterm Review

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Seminar in Engineering Analysis

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California State University Northridge

Overview

- Review last class
  - General approach
  - Diffusion equation in two space dimensions
  - Three-dimensional Laplace equation
- Review for midterm
  - Sturm-Liouville solution eigenvalues
  - General approach for PDEs
  - Transformations and superposition
  - Diffusion equation
  - Laplace equation
  - Rectangular and cylindrical coordinates

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Review 2D Diffusion

- Two-dimensional diffusion equation for u(x,y,t) $\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$t \ge 0 \quad 0 \le x \le L \quad 0 \le y \le H \quad u(x,y,0) = f(x,y)$$

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\alpha t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

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Review f(x,y) = U, a Constant

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L U\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy = \begin{cases} \frac{16}{mn\pi^2} & odd\ m\ and\ n \\ 0 & otherwise \end{cases}$$

$$\beta_n = (2n+1)\pi$$

$$\gamma_m = (2m+1)\pi$$

$$u(x,y,t) = 16U\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-\left(\frac{\beta_n^2}{L^2}+\frac{\gamma_m^2}{H^2}\right)\alpha t}}{\beta_n\gamma_m}\sin\left(\beta_n\frac{x}{L}\right)\sin\left(\gamma_m\frac{y}{H}\right)$$

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Review Nonzero Boundaries

- Sturm-Liouville eigenfunction expansions require zero boundary conditions
- For nonzero boundaries, split solution as in 1D case u(x,y,t) = v(x,y,t) + w(x,y)
  - v satisfies diffusion equation with zero boundary conditions
  - w satisfies Laplace's (and diffusion) equation with nonzero boundary conditions
  - u satisfies diffusion equation with u(x,y,t) = w(x,y) at boundaries

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Review 3D Laplace

- Use combination of separation of variables and superposition
- Start with basic solution that has homogenous boundary conditions at five surfaces and u(x,y,W) = u_W(x,y)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad 0 \le x \le L \quad 0 \le y \le H \quad 0 \le z \le W$$

$$u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,H,z) = u(x,y,0) = 0$$

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Review 3D Laplace II

- Solution of Laplace equation in x, y, z similar to diffusion solution in x, y, t
- Have hyperbolic cosine in z direction instead of exponential time decay

$$u(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sinh\left(\frac{n\pi z}{L}+\frac{m\pi z}{H}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$C_{nm}=\frac{4}{HL\sinh\left(\frac{n\pi W}{L}+\frac{m\pi W}{H}\right)}\int_0^H\int_0^L u_W(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

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Midterm Exam

- Wednesday, March 11
- Covers material on diffusion and Laplace equations (2 independent variables only)
- Includes material up to and including homework for Monday, March 2
- Open book and notes, including homework solutions and integral tables
  - Typical possible integrals: $\sin^2 ax$, $x\sin(ax)$

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Sturm-Liouville

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)+\left[q(x)+\lambda p(x)\right]y=0$$

$$k_1y(a)+k_2\left.\frac{dy}{dx}\right|_{x=a}=0$$

$$\ell_1y(b)+\ell_2\left.\frac{dy}{dx}\right|_{x=b}=0$$

- Can expand any f(x) in terms of complete set of eigenfunctions, y_m

$$f(x)=\sum_{m=0}^{\infty}a_my_m(x)\qquad a_m=\frac{(y_m,f)}{(y_m,y_m)}=\frac{\int_a^b p(x)y_m(x)f(x)dx}{\int_a^b p(x)y_m(x)y_m(x)dx}$$

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Partial Differential Equations

- Solving PDEs
  - Basic idea is to get solution to PDE as sum of eigenfunctions than can be used to represent an initial or boundary condition
    - Ability to get such a set of eigenfunctions assured if we have a Sturm Liouville problem
    - Key element of such a problem is homogenous differential equation and boundary conditions
    - Will use various transforms to get this problem
    - For diffusion equation use u(x,t) = v(x,t) + w(x)
    - For Laplace's equation use superposition

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PDE Boundaries

- Boundary conditions
  - Fixed value (first kind or Dirichlet) (*e.g.*, u = 0 at x = 0)
  - Fixed gradient (second kind or Newmann) (*e.g.*, ∂u/∂x = 0 at x = L)
  - Mixed (third kind) a $(\partial u/\partial y)_{y=H}$ + b $u_{y=H}$ = 0
  - Use transforms like u = v + w (diffusion) or superposition (Laplace) if boundary conditions do not equal zero
  - Problems in cylinder and sphere require u to be finite at r = 0

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Solving PDE Start

- Perform necessary operations if boundaries not homogenous
  - Diffusion: define u(**x**,t) = v(**x**,t) + w(**x**)
    - v satisfies diffusion equation with zero boundary conditions; w satisfies boundary
    - Can also uses this for "source term"
  - Laplace equation use superposition
    - Solution is sum of two or more solutions each of which has only one nonzero boundary
    - Each solution has consistent boundary condition kind for zero and nonzero parts

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Separation of Variables

- Not needed if existing separation of variables solution is available
  - Set variable in PDE, u, as product of two functions of one variable: u = $F(x_1)G(x_2)$
  - Substiute product into PDE and differentiate
  - Divide by original product solution to get two terms, one with F only and one with G
  - Set one term to a $-\lambda^2$ to get eigenfunctions
  - Solve resulting pair of ODEs

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Separation of Variables Result

- Starting solutions
  - Diffusion equation, rectangular geometry

$$u(x,t) = e^{-\lambda^2 \alpha t}\left[C_1 \sin(\lambda x) + C_2 \cos(\lambda x)\right]$$

  - Diffusion equation, cylindrical geometry

$$u(r,t) = e^{-\lambda^2 \alpha t}\left[C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)\right]$$

  - Diffusion equation, spherical geometry

$$u(r,t) = e^{-\lambda^2 \alpha t}\left(C_1 \frac{\sin \lambda r}{r} + C_2 \frac{\cos \lambda r}{r}\right)$$

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Separation of Variables Result II

- Rectangular Laplace starting solutions
  - Rectangular, nonzero BC at y = 0 or y = H

$$u(x,y) = \left[A \sin(\lambda x) + B \cos(\lambda x)\right]\left[C \sinh(\lambda y) + D \cosh(\lambda y)\right]$$

  - Rectangular, nonzero BC at x = 0 or x = L

$$u(x,y) = \left[A \sinh(\lambda x) + B \cosh(\lambda x)\right]\left[C \sin(\lambda y) + D \cos(\lambda y)\right]$$

  - Cylindrical, nonzero BC at z = 0 or z = H

$$u(r,z) = \left[A \sinh \lambda z + B \cosh \lambda z\right]\left[C J_0(\lambda r) + D Y_0(\lambda r)\right]$$

  - Cylindrical, nonzero BC at r = R_o or R = R_i

$$u(r,z) = \left[A \sin \lambda z + B \cos \lambda z\right]\left[C I_0(\lambda R) + D K_0(\lambda R) = 0\right]$$

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Fitting Boundary Conditions

- Start with homogenous boundary conditions (variable, gradient or sum = 0)
- Eliminate some constants in starting solutions
  - Constants will be zero or be related to each other, e.g. C = $-DI_0(\lambda R_o)/K_0(\lambda R_o)$
  - One boundary condition will lead to eigenvalues and eigenfunctions
  - Final result should be product solution with one eigenfunction and one constant

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Eigenfunction Expansions

- Most general solution is sum of infinite series of eigenfunctions, each with it's own constant: $\Sigma_n C_n F_n(\lambda\xi) G_n(\eta)$
- Use nonzero boundary condition (initial condition in diffusion equation) to fit constants *via* eigenfunction expansion

$$C_n = \frac{\int_a^b f_{\substack{boundary\\condition}}(\xi) p(\xi) F_n(\lambda_n \xi) d\xi}{G_n(B) \int_a^b p(\xi)\left[F_n(\lambda_n \xi)\right]^2 d\xi}$$

y_n is eigenfunction
λ_n is eigenvalue
B is boundary value
p is weight function

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Diffusion Equation

- For nonzero boundary conditions or additional source term, S(x), a function of x only use u(x,t) = v(x,t) + w(x)
  - v(x,t) satisfies diffusion equation with zero boundary conditions

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + S(x) \qquad au + b\frac{\partial u}{\partial x} = c \text{ at } x = 0, L$$

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} \qquad av + b\frac{\partial v}{\partial x} = 0 \text{ at } x = 0, L$$

$$\frac{d^2 w}{dx^2} + \frac{S(x)}{\alpha} = 0 \qquad aw + b\frac{dw}{dx} = 0 \text{ at } x = c, L$$

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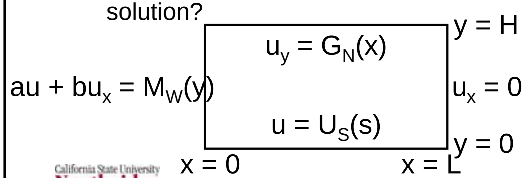
Laplace Equation Superposition

- If a region has more than one boundary with a nonhomogenous boundary condition, use superposition
- Superposition solution: the sum of two or more solutions of Laplace's equation
  - The sum of all the individual solutions satisfies the boundary conditions for the initial problem
    - Be careful in handling gradient and mixed boundary conditions

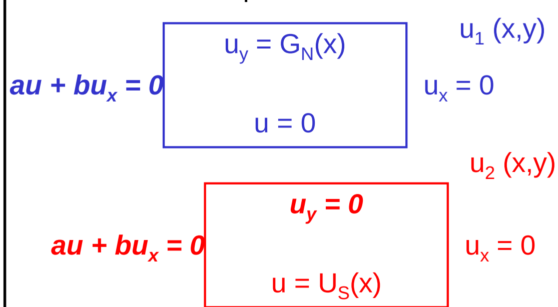
Superposition Example

- Example below has three nonzero boundaries
  - Conditions include first (fixed u), second (fixed gradient) and third (mixed)
  - What are components of superposition solution?

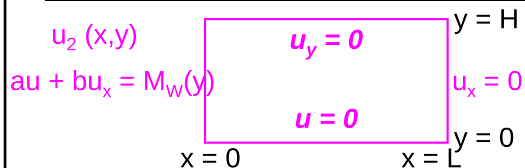


Superposition Result

- First two components



Superposition Result II



- Solution: $u(x,y) = u_1(x,y) + u_2(x.y) + u_3(x,y)$
  - Since each u_i satisfies Laplace's equation, sum must satisfy that equation
  - Can show that the sum of all three soultions satisfies original boundary conditions

Rectangular Eigenfunctions

- Start $X(x) = A\sin(\lambda x) + B\cos(\lambda x)$
  - For u = 0 at x = 0 and u = 0 at x = L
    - $X_n(x) = A\sin(n\pi x/L)$
  - For $\partial u/\partial x = 0$ at x = 0 and x = L
    - $X_n(x) = B\cos(n\pi x/L)$ (n = 0 special case)
  - For u = 0 at x = 0 and $\partial u/\partial x = 0$ at x = L
    - $X_n(x) = A\sin[(2n+1)\pi x/2L]$
  - For $\partial u/\partial x = 0$ at x = 0 and u = 0 at x = L
    - $X_n(x) = B\cos[(2n+1)\pi x/2L]$

Radial Eigenfunctions

- Start $P(r) = J_0(\lambda r) + BY_0(\lambda r)$
  - Have r = 0 in region and u = 0 at r = R
    - $P_m(r) = A_m J_0(\alpha_{m0} r/R)$ where $J_0(\alpha_{m0}) = 0$
  - Have r = 0 in region and $\partial u/\partial r = 0$ at r = R
    - $P_m(r) = A_m J_0(\alpha_{m1} r/R)$ where $J_1(\alpha_{m1}) = 0$
  - For u = 0 at $r = R_i$ and $r = R_o$
    - $P_m(r) = A_m[Y_0(\alpha_m)J_0(\alpha_m r/R_o) - J_0(\alpha_m)Y_0(\alpha_m r/R_o)]$ where α_m are the roots of $Y_0(\alpha_m)J_0(\alpha_m R_i/R_o) - J_0(\alpha_m)Y_0(\alpha_m R_i/R_o)$
  - For $\partial u/\partial r = 0$ at $r = R_i$ and $r = R_o$
    - $P_m(r) = A_m[Y_1(\alpha_m)J_0(\alpha_m r/R_o) - J_1(\alpha_m)Y_0(\alpha_m r/R_o)]$ where α_m are the roots of $Y_1(\alpha_m)J_1(\alpha_m R_i/R_o) - J_1(\alpha_m)Y_1(\alpha_m R_i/R_o L]$

Mixed Eigenfunctions

- Boundary conditions of form $-k\partial u\partial x = h(u - u_\infty)$ at x = L or x = R
- Define new variable, $v = u - u_\infty$ and write boundary condition as $\partial v\partial x + hv = 0$
- Rectangular with $\partial v\partial x = \partial u\partial x = 0$ at x = 0
  - $X_m(x) = B_m\cos(\alpha_m x/L)$ where α_m is root of $\alpha_m = (hL/k)\cot(\alpha_m)$
- For radial geometry (solid cylinder)
  - $P_m(r) = A_m J_0(\alpha_m r/R)$ where α_m is root of $(hR/k)J_0(\alpha_m) = \alpha_m J_1(\alpha_m)$

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Sample Problem

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} + Ne^{-ax} \qquad u(0,t) = u_1 \quad u(L,t) = u_2 \quad u(x,0) = f(x)$$

- Extension of homework problem
- What is first step?
  - Define u(x,t) = v(x,t) + w(x)
  - What is v(x,t)
    - Solution to diffusion equation with homogenous boundary conditions: u(0,t) = u(L,t) = 0
    - Do we know solution for v(x,t)?

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Sample Problem II

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2\alpha t}\sin(\lambda_n x) \qquad \lambda_n = \frac{n\pi}{L}$$

- Solution for v(x,t) from slide 27 of January 28 lecture
- How do we find an equation for w(x)
  - Substitute u(x,t) = v(x,t) + w(x) into diffusion equation for this problem

$$Ne^{-ax} = \frac{\partial(v+w)}{\partial t} - \alpha\frac{\partial^2(v+w)}{\partial x^2} = \frac{\partial v}{\partial t} - \alpha\frac{\partial^2 v}{\partial x^2} - \alpha\frac{\partial^2 w}{\partial x^2} = \alpha\frac{d^2 w}{dx^2}$$

- How do we find the w boundary conditions?
  - Choose them to satisfy u boundary conditions

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Sample Problem III

- Boundary conditions for w
  - u(x,t) = v(x,t) + w(x)
    - But v(0,t) = v(L,t) = 0
    - So $w(0) = u(0,t) = u_1$ and $w(L) = u(L,t) = u_2$ a
  - How do we integrate $d^2x/dx^2 = g(x)$?
    - Write $d(dw/dx)/dx = g(x)$ so $dw/dx = \int g(x)dx + C_1$

$$\frac{dw}{dx} = \int\frac{N}{\alpha}e^{-ax}dx + C_1 = -\frac{N}{a\alpha}e^{-ax} + C_1$$

$$w = \int\left(-\frac{N}{a\alpha}e^{-ax} + C_1\right)dx = \frac{N}{a^2\alpha}e^{-ax} + C_1 x + C_2$$

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Sample Problem III

- How do we find the C_1 and C_2?
  - Boundary conditions $w(0) = u_1$; $w(L) = u_2$

$$w(0) = \frac{N}{a^2\alpha}e^{-a0} + C_1 0 + C_2 = u_1 \quad\Rightarrow\quad C_2 = u_1 - \frac{N}{a^2\alpha}$$

$$w(L) = \frac{N}{a^2\alpha}e^{-aL} + C_1 L + C_2 = u_2 \quad\Rightarrow\quad u_2 - u_1 = \frac{N}{a^2\alpha}\left(e^{-aL} - 1\right) + C_1 L$$

- Still have to fit the initial condition, f(x)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2\alpha t}\sin(\lambda_n x) + w(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2\alpha 0}\sin(\lambda_n x) + w(x)$$

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Sample Problem IV

$$C_n = \frac{\int_0^L [f(x) - w(x)]\sin\frac{n\pi x}{L}dx}{\int_0^L\left[\sin\frac{n\pi x}{L}\right]^2 dx} = \frac{2}{L}\int_0^L [f(x) - w(x)]\sin\frac{n\pi x}{L}dx$$

- Can write answer from previous solutions or show details to find C_n
  - Multiply f(x) equation by $\sin(m\pi x/L)dx$
  - Integrate from 0 to L
    - All terms drop out of sum except m = n
    - Integrate $\sin^2$ and rearrange

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