Multidimensional Partial Differential Equations

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Mechanical Engineering 501B

Seminar in Engineering Analysis

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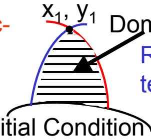
Overview

- Review last class
  - Characteristics and classification of partial differential equations
- Problems in more than two independent variables
  - Solution by separation of variables
  - Problems with multiple nonhomogenous boundary conditions
  - Solutions for rectangular geometry
  - Homework problem for cylindrical geometry

2

Review General and Hyperbolic

- Domain of dependence for $u(x_1,y_1)$
  - The area (in x-y space) whose u values affect the value of $u(x_1,y_1)$
- Region of influence of $u(x_1,y_1)$
  - The area (in x-y space) whose u values are affected by the value of $u(x_1,y_1)$
- Areas for hyperbolic equations shown below



3

Review Elliptic and Parabolic

- Imaginary characteristics for elliptic equations like Laplace and Poisson's
  - Entire solution region is both domain of dependence and region of influence
- Parabolic equations typically involve time and space as coordinates
  - Domain of dependence at x_1, t_1 is entire domain $0 \le x \le L$ and $0 \le t < t_1$
  - Region of influence at x_1, t_1 is entire region $0 \le x \le L$ and $t > t_1$

4

Review Multidimensional

- Can have equations in three space dimensions and time
- Classification as elliptic, parabolic, or hyperbolic does not apply to equations with more than two dimensions
- Coordinates can have elliptic-like, parabolic-like, and hyperbolic-like behavior in multidimensional equations
  - E. g., time is a parabolic coordinate

5

Review Multidimensional II

Laplace	Diffusion	Wave
$\nabla^2 u = -S$	$\dfrac{\partial u}{\partial t} = \alpha \nabla^2 u + S$	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + S$

Cartesian $\quad \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

Cylindrical $\quad \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial u}{\partial r} \right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2}$

Sphere $\quad \nabla^2 u = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r} \right) + \dfrac{1}{r^2 \sin^2 \phi}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\cot\phi}{r^2}\dfrac{\partial u}{\partial \phi}$

6

Review 2D Diffusion

- Two-dimensional diffusion equation for u(x,y,t) $\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$t \geq 0 \quad 0 \leq x \leq L \quad 0 \leq y \leq H \quad u(x,y,0) = f(x,y)$$

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$$

- Use separation of variable approach with all variables u(x,y,t) = X(x)Y(y)T(t)

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}$$

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Review 2D Diffusion II

- Boundary conditions give general solution as sum of all eigenfunctions

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\alpha t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

- Eigenfunction expansion for t = 0

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

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Look at f(x,y) = U, a Constant

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$ General Result

- When we substitute f(x,y) – U, we can separate the x and y integrations

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L U\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy =$$

$$\frac{4U}{HL}\left[\int_0^H \sin\left(\frac{m\pi y}{H}\right)dy\right]\left[\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right]$$

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Look at f(x,y) = constant U II

- Both integrals are effectively the same
  - for z = x, W = L and for z = y, W = H

$$\int_0^W \sin\left(\frac{p\pi z}{W}\right)dx = -\frac{W}{p\pi}\left[\cos\left(\frac{p\pi z}{W}\right)\right]_0^W =$$

$$-\frac{W}{n\pi}\left[\cos(n\pi) - 1\right] = \begin{cases} \frac{2W}{n\pi} & odd\ n \\ 0 & even\ n \end{cases}$$

$$C_{nm} = \frac{4U}{HL}\left[\int_0^H \sin\left(\frac{m\pi y}{H}\right)dy\right]\left[\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right] = \frac{4U}{HL}\frac{2H}{m\pi}\frac{2L}{n\pi} = \frac{16}{mn\pi^2}$$

California State University Northridge C_{nm} = 0 zero for even m or n 10

Result for f(x,y) = U, a Constant

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\alpha t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

- Replace n by 2n + 1 and m by 2m + 1 to get odd indices only and define new parameters β_n and γ_m as follows

$$\beta_n = (2n+1)\pi \qquad \gamma_m = (2m+1)\pi$$

$$C_{nm} = \frac{16U}{n_{old} m_{old}\pi^2} = \frac{16U}{(2n+1)(2m+1)\pi^2} = \frac{16U}{\beta_n\gamma_m}$$

$$u(x,y,t) = 16U\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-\left(\frac{\beta_n^2}{L^2}+\frac{\gamma_m^2}{H^2}\right)\alpha t}}{\beta_n\gamma_m}\sin\left(\beta_n\frac{x}{L}\right)\sin\left(\gamma_m\frac{y}{H}\right)$$

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Dimensionless Parameters

- Modify exponential argument as follows

$$e^{-\left(\frac{\beta_n^2}{L^2}+\frac{\gamma_m^2}{H^2}\right)\alpha t} = e^{-\left(\beta_n^2+\gamma_m^2\frac{L^2}{H^2}\right)\frac{\alpha t}{L^2}}$$

- Substitute into u(x,y,t) and divide by U

$$\frac{u(x,y,t)}{U} = 16\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-\left(\beta_n^2+\frac{\gamma_m^2 L^2}{H^2}\right)\frac{\alpha t}{L^2}}}{\beta_n\gamma_m}\sin\left(\beta_n\frac{x}{L}\right)\sin\left(\gamma_m\frac{y}{H}\right)$$

- Where $\beta_n = (2n+1)\pi \quad \gamma_m = (2m+1)\pi$

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Important Parameters

$$\frac{u(x,y,t)}{U} = 16\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-\left(\beta_n^2+\frac{\gamma_m^2 L^2}{H^2}\right)\frac{\alpha t}{L^2}}}{\beta_n\gamma_m}\sin\left(\beta_n\frac{x}{L}\right)\sin\left(\gamma_m\frac{y}{H}\right)$$

- Result shows that u/U is a function of x/L, y/H, L/H, and $\alpha t/L^2$
- Can simplify double summation in this case by splitting exponential term

$$e^{-\left(\beta_n^2+\frac{\gamma_m^2 L^2}{H^2}\right)\frac{\alpha t}{L^2}} = e^{-\beta_n^2\frac{\alpha t}{L^2}}e^{-\frac{\gamma_m^2 L^2}{H^2}\frac{\alpha t}{L^2}} = e^{-\beta_n^2\frac{\alpha t}{L^2}}e^{-\gamma_m^2\frac{\alpha t}{H^2}}$$

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13

Product Solution

- Can now separate n and m sums

$$\frac{u(x,y,t)}{U} = 16\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-\beta_n^2\frac{\alpha t}{L^2}}}{\beta_n}\sin\left(\beta_n\frac{x}{L}\right)\frac{e^{-\gamma_m^2\frac{\alpha t}{H^2}}}{\gamma_m}\sin\left(\gamma_m\frac{y}{H}\right)$$

$$= 16\sum_{n=0}^{\infty}\frac{e^{-\beta_n^2\frac{\alpha t}{L^2}}}{\beta_n}\sin\left(\beta_n\frac{x}{L}\right)\sum_{m=0}^{\infty}\frac{e^{-\gamma_m^2\frac{\alpha t}{H^2}}}{\gamma_m}\sin\left(\gamma_m\frac{y}{H}\right)$$

$$= \left[4\sum_{n=0}^{\infty}\frac{e^{-\beta_n^2\frac{\alpha t}{L^2}}}{\beta_n}\sin\left(\beta_n\frac{x}{L}\right)\right]\left[4\sum_{m=0}^{\infty}\frac{e^{-\gamma_m^2\frac{\alpha t}{H^2}}}{\gamma_m}\sin\left(\gamma_m\frac{y}{H}\right)\right]$$

Product of one-dimen-sional solutions

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14

Nonzero Boundaries

- Sturm-Liouville eigenfunction expan-sions require zero boundary conditions
- For nonzero boundaries, split solution as in 1D case u(x,y,t) = v(x,y,t) + w(x,y)
  - v satisfies diffusion equation with zero boundary conditions
  - w satisfies Laplace's (and diffusion) equation with nonzero boundary conditions
  - u satisfies diffusion equation with u(x,y,t) = w(x,y) at boundaries

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15

Nonzero Boundaries II

- Solve Laplace equation for w (with superposition if required)
- Solution for v is same as previous solution for u with zero boundaries

$$v(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

- Initial condition for u found from u(x,y,t) = v(x,y,t) + w(x,y) by setting t = 0

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16

Nonzero Boundaries III

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)+\boxed{w(x,y)}$$

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)+\boxed{w(x,y)}$$

- Use eigenfunction expansion to determine C_{nm} to satisfy initial condition
- Must include w(x,y) in computing C_{nm} similar to case with one space dimension

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17

Nonzero Boundaries IV

- Same eigenfunction expansion used for zero boundary problem gives

$$C_{nm} = \frac{4}{HL}\int_0^H\int_0^L\left(f(x,y)-w(x,y)\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

- Simplest example is case where we have
  - Constant initial potential, f(x,y) = U_0
  - Boundary potential of zero on three sides
  - Constant potential, U_{side}, on fourth side
  - Know Laplace equation solution for w(x,y)

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18

Problem and Solution Start

- Diffusion equation for u(x,y,t)

$$\frac{1}{\alpha}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \quad t\ge 0 \quad 0\le x\le L \quad 0\le y\le H$$

$$u(x,y,0)=U_0 \quad u(x,0,t)=u(0,y,t)=u(L,y,t)=0 \quad u(x,H,t)=U_N$$

- Solution v(x,y,t) + w(x,y) where v(x,y,t) just found

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)+w(x,y)$$

- Know w(x,y) from previous solutions of Laplace's equation

Using the Steady Solution

- Laplace equation solution for w(x,0) = w(0,y) = w(L,y) = 0 and w(x,H) = U_N

$$w(x,y)=\frac{4U_N}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- Must use this solution in C_{nm} equation

$$C_{nm}=\frac{4}{HL}\int_0^H\int_0^L\left(f(x,y)-w(x,y)\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

Getting the Answer

- Have to perform integration for C_{nm}
  - Not difficult, but a lot of algebra and potential for error
  - First part of integral for f(x,y) = U_0, a constant, available from previous example
  - Second part with 2D, steady-state solution requires most work
  - MATLAB does not give good result for integral of sin(ay)sinh(by)
  - Integration details at end of presentation

Getting the Answer II

- Laplace equation solution for w(x,y)

$$w(x,y)=\frac{4U_N}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- Split equation for C_{nm} into two parts

f(x,y) = U_0

$$C_{nm}=\frac{4}{HL}\int_0^H\int_0^L U_0\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy \quad = \frac{16U_0}{nm\pi^2}$$

zero unless both n and m odd

$$-\frac{4}{HL}\int_0^H\int_0^L w(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dxdy$$

Result for C_{nm}

- Integration shows C_{nm} = 0 for even n

$$C_{nm}=\begin{cases}\frac{16U_0}{nm\pi^2}+\frac{8(-1)^m U_N m}{nHL}\frac{1}{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2} & m\ odd,\ n\ odd\\ 0 & n\ even,\ any\ m\\ \frac{8(-1)^m U_N m}{nHL}\frac{1}{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2} & m\ even,\ n\ odd\end{cases}$$

Get Dimensionless Solution

- Divide solution by U_N and factor out $1/L^2$ term in exponential argument

$$u(x,y,t)=\sum_{n=1,3,5\ldots}^{\infty}\sum_{m=1}^{\infty}C_{nm}e^{-\left[\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)+w(x,y)$$

$$\frac{u(x,y,t)}{U_N}=\sum_{n=1,3,5\ldots}^{\infty}\sum_{m=1}^{\infty}\frac{C_{nm}}{U_N}e^{-\left[n^2+\left(\frac{mL}{H}\right)^2\right]\pi^2\frac{\alpha t}{L^2}}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)+\frac{w(x,y)}{U_N}$$

$$w(x,y)=\frac{4U_N}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi Hy}{LH}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

Equations for w(x,y) and C_{nm}

- Divide C_{nm} by U_N and factor out π^2/L^2 denominator of second fraction

$$\frac{C_{nm}}{U_N} = \begin{cases} \dfrac{16U_0}{nm\pi^2 U_N} + \dfrac{8(-1)^m m}{\pi^2 n \dfrac{H}{L}} \dfrac{1}{n^2 + \left(\dfrac{mL}{H}\right)^2} & m \text{ odd} \\[2ex] \dfrac{8(-1)^m m}{\pi^2 n \dfrac{H}{L}} \dfrac{1}{n^2 + \left(\dfrac{mL}{H}\right)^2} & m \text{ even} \end{cases}$$

- Solution u/U_N depends on x/L, y/H, $\alpha t/L^2$, U_0/U_N and H/L

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Results for $u(x,y,t)/U_N$

- Solution u/U_N depends on x/L, y/H, $\alpha t/L^2$, U_0/U_N and H/L
- View solution as u/U_N contours as a function of x/L and y/H coordinates
- Select $H/L = 1$ and $U_0/U_N = 2$
  - Boundary $u = 0$ but at $y = H$ where $u/U_N = 1$
- Obtain one plot for a given value of $\alpha t/L^2$
- Sequential plots show evolution u/U_N from initial conditions to steady state

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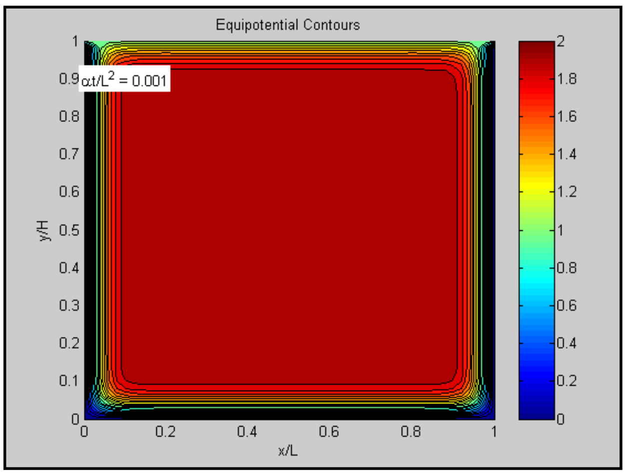

Three-Dimensional Laplace

- Use combination of separation of variables and superposition
- Start with basic solution that has homogenous boundary conditions at five surfaces and $u(x,y,W) = u_W(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad 0 \le x \le L \quad 0 \le y \le H \quad 0 \le z \le W$$

$$u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,H,z) = u(x,y,0) = 0$$

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Three-Dimensional Laplace II

- Solution of Laplace equation in x, y, z similar to diffusion solution in x, y, t
- Have hyperbolic cosine in z direction instead of exponential time decay

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sinh\left(\frac{n\pi z}{L} + \frac{m\pi z}{H}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

$$C_{nm} = \frac{4}{HL \sinh\left(\dfrac{n\pi W}{L} + \dfrac{m\pi W}{H}\right)} \int_0^H \int_0^L u_W(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\,dy$$

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Homework for March 16

- Kreyszig, page 586, problems 24 – 27
- Two dimensional wave equation in a cylinder $0 \le r \le R$, $0 \le \theta \le 2\pi$, $t \ge 0$

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

- Find $u(r,\theta,t)$ by separation of variables
- Periodic boundary condition in θ direction: $u(r,0,t) = u(r,2\pi,t)$

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Reminder and Additional Material

- Midterm next Wednesday (March 11)
  - Covers material on homework assignments submitted up to and including last Monday (March 2)
  - Will be open notes and book, including integral tables, but no computers
- Charts 32 to 34 show start of homework problem for Monday
- Charts 35 to 43 show details of finding the C_{nm} coefficients on chart 23 for diffusion in two space dimensions

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Work on Homework Problem

- Page 547, problem 12 – solve the wave equation for 0 ≤ x ≤ L = π and t ≥ 0 with boundary and initial conditions shown

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad c = 1 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0.01x & 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$
$$u(0,t) = u(L,t) = 0 \qquad u(x,0) = 0$$

- Start with separation of variables result

$$u(x,t) = T(t)X(x) =$$
$$[A\sin(\lambda ct) + B\cos(\lambda ct)][C\sin(\lambda x) + D\cos(\lambda x)]$$

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Work on Homework Problem

- Have already seen that boundary conditions that u(0,t) = u(L,t) = 0 give following solution

$$u(x,t) = \left[A_n \sin\frac{n\pi ct}{L} + B_n\cos\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}$$

- Coefficients depend on initial displacement, u(x,0) = f(x) and velocity, u_x(x,0) = g(x)

$$A_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

California State University Northridge For f(x) = 0, all B_n = 0 33

Work on Homework Problem

- Solution for B_m = 0

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n ct)\sin(\lambda_n x)$$

- Use function given for g(x) to get A_n

$$A_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{2}{n\pi c}\int_0^{\frac{\pi}{2}} 0.01x\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{n\pi c}\int_{\frac{\pi}{2}}^{\pi} 0.01(\pi - x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- Set L = π and c = 1

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2D Diffusion Details

- Solution is given by

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\alpha t}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) + w(x,y)$$

- C_{nm} is found from initial condition f(x,y) and steady-state solution, w(x,y)

$$C_{pq} = \frac{4}{HL}\int_0^H\int_0^L (f(x,y) - w(x,y))\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

- Apply to f(x,y) = U_0, a constant, three sides at zero and one side at U_S

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2D Diffusion Details II

- Laplace equation solution for w(x,y)

$$w(x,y) = \frac{4U_S}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- Split equation for C_{pq} into two parts

$$C_{pq} = \frac{4}{HL}\int_0^H\int_0^L U_0\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy = \frac{16U_0}{pq\pi^2}$$
$$-\frac{4}{HL}\int_0^H\int_0^L w(x,y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

zero unless both p and q odd

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Get Second Part of C_{pq} Integral

$$\frac{4}{HL}\int_0^H\int_0^L w(x,y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy =$$

$$\frac{4}{HL}\int_0^H\int_0^L \frac{4U_S}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

$$=\frac{16U_S}{HL\pi}\sum_{n=0}^{\infty}\left\{\frac{1}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}\left[\int_0^L \sin\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right)dx\right]\bullet \int_0^H \sinh\left(\frac{(2n+1)\pi y}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dy\right\}$$

Northridge 37

$\int_0^H \sinh ay \sin by dy$ Details

- Apply integral table formula from 0 to H

$$\int \sinh ax \sin bx dx = \frac{-b\sinh ax\cos bx + a\cosh ax \sin bx}{a^2+b^2}$$

$$\int_0^H \sinh ay \sin by dy = \left[\frac{-b\sinh ay\cos by + a\cosh ay \sin by}{a^2+b^2}\right]_0^H$$

$$= \frac{-b\sinh aH\cos bH + b\sinh 0\cos 0 + a\cosh aH \sin bH - a\cosh 0 \sin 0}{a^2+b^2}$$

$$\int_0^H \sinh ay \sin by dy = \frac{-b\sinh aH\cos bH + a\cosh aH \sin bH}{a^2+b^2}$$

Northridge 38

$\int_0^H \sinh ay \sin by dy$ Details II

- Substitute a = (2n + 1)π/L and b = qπ/H

$$\int_0^H \sinh\left(\frac{(2n+1)\pi y}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dy = \frac{-\frac{q\pi}{L}\sinh\left(\frac{(2n+1)\pi H}{L}\right)\cos\left(\frac{q\pi H}{H}\right) + \frac{(2n+1)\pi}{L}\cosh\left(\frac{(2n+1)\pi H}{L}\right)\sin\left(\frac{q\pi H}{H}\right)}{\left(\frac{(2n+1)\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2}$$

(-1)^q 0

$$\int_0^H \sinh\left(\frac{(2n+1)\pi y}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dy = \frac{(-1)^{q+1}\frac{q\pi}{L}\sinh\left(\frac{(2n+1)\pi H}{L}\right)}{\left(\frac{(2n+1)\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2}$$

Northridge 39

Results for Two Integrals

$$\int_0^L \sin\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right)dx = \begin{cases}\frac{L}{2} & p=2n+1\\ 0 & otherwise\end{cases}$$

$$\int_0^H \sinh\left(\frac{(2n+1)\pi y}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dy = \frac{(-1)^{q+1}\frac{q\pi}{L}\sinh\left(\frac{(2n+1)\pi H}{L}\right)}{\left(\frac{(2n+1)\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2}$$

- Two results are multiplied together giving a zero unless 2n+1 = p
- Previous result: p an odd integer

Northridge 40

Review Start of C_{pq} for f(x,y) = U_0

$$C_{pq} = \frac{4}{HL}\int_0^H\int_0^L (f(x,y)-w(x,y))\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

$$\frac{4}{HL}\int_0^H\int_0^L U_0\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy = \frac{16U_0}{pq\pi^2} \quad \begin{matrix}odd\ \ p\ and\\ q\ only\end{matrix}$$

$$w(x,y) = \frac{4U_S}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

Northridge 41

Combine Previous Results

$$C_{pq} = \frac{16U_0}{pq\pi^2} - \frac{4}{HL}\int_0^H\int_0^L w(x,y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

zero unless both p and q odd

$$C_{pq} = \frac{16U_0}{pq\pi^2} - \frac{16U_S}{HL\pi}\sum_{n=0}^{\infty}\left\{\frac{1}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}\bullet \left[\int_0^L \sin\left(\frac{(2n+1)\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right)dx\right]\int_0^H \sinh\left(\frac{(2n+1)\pi y}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dy\right\}$$

$$= \begin{cases}\frac{L}{2} & p=2n+1\\ 0 & otherwise\end{cases} \qquad = \frac{(-1)^{q+1}\frac{q\pi}{L}\sinh\left(\frac{(2n+1)\pi H}{L}\right)}{\left(\frac{(2n+1)\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2}$$

Northridge 42

Result for C_{pq} (zero for even p)

$$C_{pq} = \frac{16U_0}{pq\pi^2} - \frac{16U_S}{HL\pi}\frac{1}{p\sinh\left(\frac{p\pi H}{L}\right)}\frac{L}{2}\frac{(-1)^{q+1}\frac{q\pi}{L}\sinh\left(\frac{p\pi H}{L}\right)}{\left(\frac{p\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2}$$

8

$-(-1)^{q+1} = (-1)^q$

$$C_{pq} = \begin{cases} \frac{16U_0}{pq\pi^2} + \frac{8(-1)^q U_S q}{pHL}\frac{1}{\left(\frac{p\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2} & q\ odd \\ \frac{8(-1)^q U_S q}{pHL}\frac{1}{\left(\frac{p\pi}{L}\right)^2+\left(\frac{q\pi}{H}\right)^2} & q\ even \end{cases}$$

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43