Classification of PDEs and Multidimensional PDEs

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Mechanical Engineering 501B

Seminar in Engineering Analysis

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California State University Northridge

Overview

- Review last class
  - – Wave equation solutions by separation of variables and D'Alembert approach
- Characteristics and classification of partial differential equations
  - – General analysis
  - – Parabolic equations
  - – Elliptic equations
  - – Hyperbolic equations
- Solving a wave equation problem

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Midterm Exam

- Wednesday, March 11
- Covers material on diffusion and Laplace equations
- Includes material up to and including tonight's February 23 lecture and homework for Monday, March 2
- Open book and notes, including homework solutions
- Focus on working with existing solutions

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Review Wave Equation

- Usual assumption u(x,t) = X(x)T(t)

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2$$

Result is function of t equal to function of x

$$u(x,t) = T(t)X(x) = \left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

- Use above solution as starting point
  - – Boundary conditions at x = 0 and x = L
  - – Initial conditions on u and ∂u/∂t at x = 0

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Review General Solution

- Solution for u(x,t) with initial and boundary conditions
  - – u(x,0) = f(x); $\partial u/\partial x|_0$ = g(x)
  - – u(0,t) = u(L,t) = 0

$$\boxed{\begin{gathered}\frac{\partial^2 u}{\partial^2 t} = c^2\frac{\partial^2 u}{\partial x^2}\\ 0 \le x \le L,\, t \ge 0\\ c \textit{ is wave speed}\end{gathered}}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$A_m = \frac{2}{m\pi}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

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Review Use of Trig Identities

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \overset{\textbf{sin y}}{\sin\left(\frac{n\pi ct}{L}\right)} + B_n\overset{\textbf{cos y}}{\cos\left(\frac{n\pi ct}{L}\right)}\right]\overset{\textbf{sin z}}{\sin\left(\frac{n\pi x}{L}\right)}$$

- 2 sin z cos y = sin(z + y) + sin(z – y)
- 2 sin z sin y = cos(z – y) – cos(z + y)

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left\{A_n\left[\cos\left(\frac{n\pi(x-ct)}{L}\right) - \cos\left(\frac{n\pi(x+ct)}{L}\right)\right] + B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right]\right\}$$

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Review D'Alembert Solution

- Wave phenomena: u(x,t) is wave amplitude varying with space, x, and time, t
- c is wave speed
- Over any x region and t ≥ 0

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- D'Alembert solution using $\xi = x + ct$ and $\eta = x - ct$, $u(x,0) = f(x)$ and $\partial u/\partial x|_{t=0} = g(x)$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(v)\,dv$$

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Review D'Alambert II

- We see that the solution obtained by separation of variables agrees with the D'Alambert solution for one case
- Solution shows propagation of wave shapes without damping
  - Look at meaning of f(x + ct) and f(x – ct)
    - As time increases f(x + ct) retains the shape of the initial condition and moves to the left
    - Similarly f(x – ct) retains its shape and moves to the right

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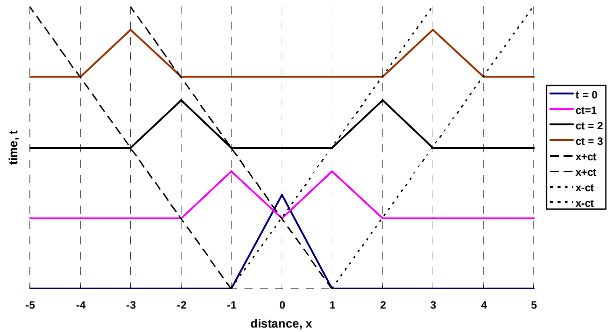
Review Wave Propagation



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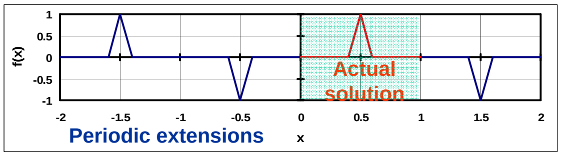
Review Boundaries

- With g(x) = 0 solution is Fourier sine series which is periodic, odd function

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$



$$B_m = \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right]$$

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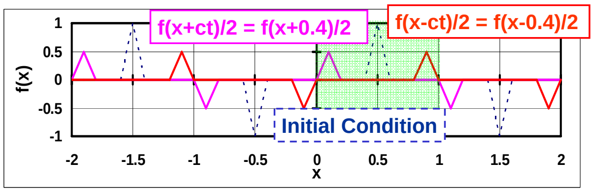
Review Time Evolution

- Look at evolution when ct = 0.4

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$



- For larger values of x ± ct, periodic extensions move into 0 ≤ x ≤ L = 1

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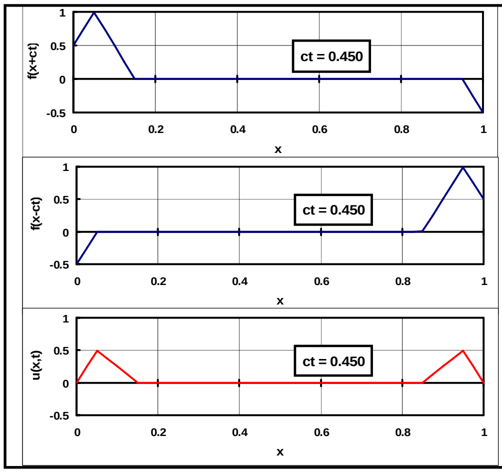


Phase behavior of sine function causes initial wave form to be reflected at boundaries

Review Characteristics

- Used to classify second-order PDEs

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

- Analysis gives slope of “characteristics”

$$\frac{dy}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

- Characteristics slopes gives region of influence and domain of dependence

Review Classification of PDEs

- The general second-order PDE in two variables is classified as follows
  - If $B^2 - 4AC < 0$ the PDE is called elliptic and has no real characteristic directions
  - If $B^2 - 4AC = 0$ the PDE is parabolic and has one repeated characteristic direction
  - If $B^2 - 4AC > 0$ the PDE is hyperbolic and has two real characteristic directions

$$\frac{dy}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-D=0$$

Laplace/Poisson/Helmholtz equations ($A = C = 1$, $B = 0$, $B^2 - 4AC < 0$) are elliptic (no real characteristics)

Diffusion equation ($A = \alpha > 0$, $B = C = 0$, $B^2 - 4AC = 0$) is parabolic (one characteristic)

$$\alpha\frac{\partial^2 u}{\partial x^2}+D=0$$

Wave equation ($B^2 - 4AC > 0$) is hyperbolic (two real characteristics)

Wave Equation Characteristics

- Compute characteristic directions

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

Wave equation: $A = 1$; $B = 0$; $C = -c^2$; $B^2 - 4AC = c^2 > 0$

$$A\frac{\partial^2 u}{\partial t^2}+B\frac{\partial^2 u}{\partial t\partial x}+C\frac{\partial^2 u}{\partial x^2}+D\left(t,x,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)=0$$

$$\frac{dx}{dt}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{-0\pm\sqrt{0^2-4(1)(-c^2)}}{2(1)}=\pm c$$

Wave Equation Characteristics II

- Compute characteristic directions with order of variables reversed

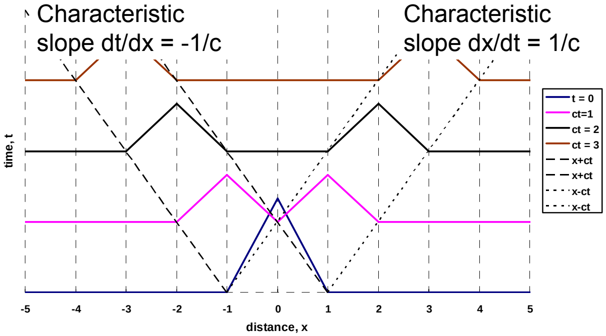
$$c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0$$

Wave equation: $A = c^2$; $B = 0$; $C = -1$; $B^2 - 4AC = c^2 > 0$

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial t}+C\frac{\partial^2 u}{\partial t^2}+D\left(t,x,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)=0$$

$$\frac{dt}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\frac{-0\pm\sqrt{0^2-4(1)(-c^2)}}{2\left(-c^2\right)}=\pm\frac{1}{c}$$

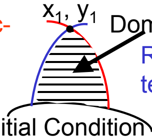
Review Wave Propagation


Characteristic slope dt/dx = -1/c

Characteristic slope dx/dt = 1/c

Behavior of Equation Types

- Domain of dependence for $u(x_1,y_1)$
  - The area (in x-y space) whose u values affect the value of $u(x_1,y_1)$
- Region of influence of $u(x_1,y_1)$
  - The area (in x-y space) whose u values are affected by the value of $u(x_1,y_1)$
- Importance for specifying boundary conditions and for numerical solutions


Left-running characteristic (dy/dx < 0)

x_1, y_1

Domain of dependence

Right-running characteristic (dy/dx > 0)

Initial Condition Curve

Hyperbolic Equations

- Domain of dependence shown on previous chart
- Region of influence is region of characteristics leaving x_1, y_1
- Conditions outside domain of dependence should not affect solution
- Important point for numerical algorithms which can violate this principle for inappropriate choices of step sizes

Elliptic PDEs

- Imaginary characteristics for elliptic equations like Laplace and Poisson’s
- Entire solution region is both domain of dependence and region of influence
  - Must have closed boundaries
- This means that any change in any boundary condition can affect the solution at any point in the region
  - Effects may be small far from boundary, but will be present

Parabolic PDEs

- Parabolic equations typically involve time and space as coordinates
- Consider region $0 \le x \le L$ and $t > 0$
  - Domain of dependence at any point x_1, t_1 is entire domain at previous times: $0 \le x \le L$ and $0 \le t < t_1$
    - Any change in initial conditions or boundary conditions for $t < t_1$ will change solution here
  - Region of influence at x_1, t_1 is entire region for future times $0 \le x \le L$ and $t > t_1$

Example Question

- You are solving the diffusion equation in the region $0 \le x \le L$ and $t > 0$ with an initial condition $u(x,0) = f(x)$ and the following boundary conditions
  - $t < 12$ s: $u(0,t) = u(L,t) = 0$
  - $t \ge 12$ s: $u(0,t) = u(L,t) = a = 1$
- If you have a solution to this problem for a = 1, how does the solution change for t < 12 s, if you set a = 2?

Work on Homework Problem

- Page 547, problem 12 – solve the wave equation for $0 \le x \le L = \pi$ and $t \ge 0$ with boundary and initial conditions shown

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad c = 1 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0.01x & 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

$$u(0,t) = u(L,t) = 0 \qquad u(x,0) = 0$$

- Start with separation of variables result

$$u(x,t) = T(t)X(x) =$$
$$\left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

Work on Homework Problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad c = 1 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0.01x & 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

$$u(0,t) = u(L,t) = 0 \qquad u(x,0) = 0$$

$$u(x,t) = T(t)X(x) =$$

$$[A\sin(\lambda ct) + B\cos(\lambda ct)][C\sin(\lambda x) + D\cos(\lambda x)]$$

- For u(0,t) = 0 we must have D = 0
- For u(L,t) = 0 we must have λL = nπ (n an integer)

Work on Homework Problem

$$u(x,t) = [A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)]\sin(\lambda_n x)$$

- From solution for u = 0 at x = 0 and x = L, we have for u(x,0) = f(x), $u_t(x,0)$ = g(x)

$$A_m = \frac{2}{m\pi c}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

- In this problem f(x) = u(x,0) = 0 so we have all B_m = 0

Work on Homework Problem

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n ct)\sin(\lambda_n x)$$

- Use function given for g(x) to get A_m

$$A_m = \frac{2}{m\pi c}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$A_m = \frac{2}{m\pi c}\int_0^{\frac{\pi}{2}} 0.01x\sin\left(\frac{m\pi x}{L}\right)dx + \frac{2}{m\pi c}\int_{\frac{\pi}{2}}^{\pi} 0.01(\pi - x)\sin\left(\frac{m\pi x}{L}\right)dx$$

- Set L = π and c = 1

Multidimensional Equations

- Can have equations in three space dimensions and time
- Classification as elliptic, parabolic, or hyperbolic does not apply to equations with more than two dimensions
- Coordinates can have elliptic-like, parabolic-like, and hyperbolic-like behavior in multidimensional equations
  - – E. g., time is a parabolic coordinate

Multidimensional Laplace

- General Laplace equation for three dimensions

$$\nabla^2 u = 0$$

Cartesian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Cylindrical $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

Sphere $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$

Multidimensional Diffusion

- General diffusion equation for three dimensions

$$\frac{\partial u}{\partial t} = \alpha\nabla^2 u$$

Cartesian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Cylindrical $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

Sphere $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$

Multidimensional Wave Equation

- General wave equation for three dimensions $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

$$\textit{Cartesian} \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\textit{Cylindrical} \qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\textit{Sphere}\ \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$$

Physical Source Terms

- Poisson Equation is Laplace's Equation with a source term, S $\nabla^2 u = -S$
- Source term can depend on coordinates (x,y,z), (r,θ,z), or (r,θ,ø) and u
- Can have similar terms in diffusion equation and in wave equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + S \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + S$$

Multidimensional Solutions

- Two-dimensional diffusion equation for u(x,y,t) $\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$t \ge 0 \quad 0 \le x \le L \quad 0 \le y \le H \quad u(x,y,0) = f(x,y)$$

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$$

- Use separation of variable approach with all variables u(x,y,t) = X(x)Y(y)T(t)

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}$$

Multidimensional Solutions II

- Separation of variables works, each term must equal a constant

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2 \qquad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\kappa^2$$

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\lambda^2 - \kappa^2$$

Multidimensional PDEs II

- Have 3 ODEs with known solutions

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2 \qquad X(x) = A\sin\lambda x + B\cos\lambda x$$

$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\kappa^2 \qquad Y(y) = C\sin\kappa y + D\cos\kappa y$$

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -\lambda^2 - \kappa^2 \qquad T(t) = Ee^{-(\lambda^2+\kappa^2)\alpha t}$$

Apply Boundary Conditions

- u(0,y,t) = u(L,y,t) = 0 => X(0) = X(L) = 0
- X(0) = 0 gives B = 0 and X(L) = 0 gives λ = nπ/L so X(x) = Asin(nπx/L)
- u(x,0,t) = u(x,H,t) = 0 => Y(0) = Y(H) = 0
- Y(0) = 0 gives D = 0 and Y(H) = 0 gives κ = mπ/H so Y(y) = Csin(mπy/H)
- $u(x,y,t) = (ACE)e^{-\alpha t[(n\pi x/L)^2 + (m\pi y/H)^2]} \bullet \sin(n\pi x/L) \sin(m\pi y/H)$ (eigenfunction)
- Define constant as ACE = C_{nm}

Initial Condition

- Get general solution as sum of all eigenfunctions and get initial condition

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} e^{-\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\alpha t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

- Multiply by [sin(pπx/L) sin(qπy/H)]dxdy and integrate for 0 ≤ x ≤ L and 0 ≤ y ≤ H

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Initial Condition Gives C_{mn}

- Use orthogonality of sine and previous results for $\sin^2$ integrals to get

$$\int_0^H\int_0^L f(x,y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy =$$

$$\int_0^H\int_0^L \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

$$= C_{pq}\left[\int_0^H \sin^2\left(\frac{q\pi y}{H}\right)dy\right]\int_0^L \sin^2\left(\frac{p\pi x}{L}\right)dx = C_{pq}\frac{H}{2}\frac{L}{2}$$

$$C_{pq} = \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{q\pi y}{H}\right)dxdy$$

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