Wave Equation and Introduction to Classification of PDEs

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Seminar in Engineering Analysis

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California State University Northridge

Overview

- Review last class
  - Wave equation solutions by separation of variables and D'Alembert approach
- Wave equation solution with boundaries
- Characteristics and classification of partial differential equations
  - General analysis
  - Parabolic equations
  - Elliptic equations
  - Hyperbolic equations

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Course Items

- Notes on wave equation on web site
- Midterm – Wednesday, March 11
  - Covers material on diffusion and Laplace equations
  - Includes material up to and including lecture and homework for March 2
  - Open textbook and notes, including homework solutions
  - Use existing solutions to answer questions

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Review Gradients

- Gradients of Laplace equation solutions often proportional to flux terms
  - Heat flux and temperature gradient
  - Diffusion flux and mass fraction gradient
  - Velocity and velocity potential in ideal flow
  - In constant potential plot, lines perpen-dicular to the potential are flux lines

$$grad\ f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

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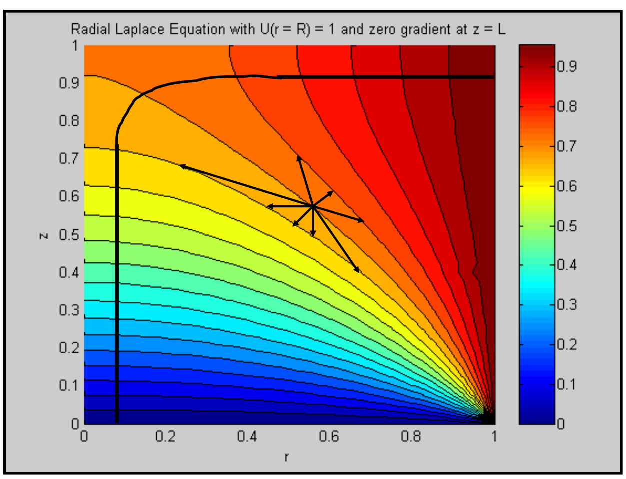

Review Interpretation of $\nabla^2 u = 0$

- When **v** = -k grad u is a flux that is the gradient of a scalar, Laplace's equation for u says that the net inflow of v is zero

$$\iiint_{\substack{Enclosed\\Volume}} \nabla^2 u dV = -\frac{1}{k}\iint_{Surface} \mathbf{v}\cdot\mathbf{n} dA = 0$$

- Example of this result shown last week
- Result applies to any problem in any geometry with Laplace's equation

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Review Complex Variable

- Cauchy-Riemann conditions

$$\text{If} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{then} \quad \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

- Equivalent to Laplace equation
  - Function u(x,y) that satisfies Laplace equation in two dimensions, has associated function v(x,y) that satisfies Laplace
  - Lines of u(x,y) and v(x,y) are perpendicular
  - Typically if u is a potential (e.g, temperature, v is a corresponding flux)

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Review Additional Results

- Cauchy theorem for complex integration shows Laplace equation solutions
  - Have maximum and minimum on boundary
  - If boundary is a constant at all points then solution is the same constant in region
  - Dirichlet problem has unique solution
  - Neumann problem does not
- Kreyszig section 18.6 has proofs

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Review Wave Equation

- Wave phenomena: u(x,t) is wave amplitude varying with space, x, and time, t
- c is wave speed

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Can solve by usual separation of variables technique
- Also have D'Alambert solution with arbitrary functions F and G with coordinates ξ = x + ct and η = x – ct

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Review Separation of Variables

- Usual assumption u(x,t) = X(x)T(t)

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2$$

Result is function of t equal to function of x

$$u(x,t) = T(t)X(x) = \left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

- Use above solution as starting point
  - Boundary conditions at x = 0 and x = L
  - Initial conditions on u and ∂u/∂t at x = 0

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Review Separation of Variables

- Solution for u(x,t) with initial and boundary conditions
  - u(x,0) = f(x); $\partial u/\partial x|_0$ = g(x)
  - u(0,t) = u(L,t) = 0

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}$$
$$0 \le x \le L, t \ge 0$$
$$c \text{ is wave speed}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$A_m = \frac{2}{m\pi}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

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General Solution

- Substitute A_m and B_m from equations just found and substitute into previous solution

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

- Examine case where g(x) = 0 so A_n = 0

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

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General Solution for g(x) = 0

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

- From trig identities for sin(x ± y)
  - sin(x + y) = sin x cos y + sin y cos x
  - sin(x – y) = sin x cos y – sin y cos x
  - sin(x + y) + sin(x – y) = 2 sin x cos y

$$u(x,t)=\frac{1}{2}\sum_{n=1}^{\infty}B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right)+\sin\left(\frac{n\pi(x-ct)}{L}\right)\right]$$

Similar Solution for f(x) = 0

$$u(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

- From trig identities for cos(x ± y)
  - cos(x + y) = cos x cos y – sin y sin x
  - cos(x – y) = cos x cos y + sin y sin x
  - cos(x – y) – cos(x + y) = 2 sin x sin y

$$u(x,t)=\frac{1}{2}\sum_{n=1}^{\infty}A_n\left[\cos\left(\frac{n\pi(x-ct)}{L}\right)-\cos\left(\frac{n\pi(x+ct)}{L}\right)\right]$$

D'Alambert Solution

- Wave phenomena: u(x,t) is wave amplitude varying with space, x, and time, t
- c is wave speed

$$\frac{\partial^2u}{\partial t^2}=c^2\frac{\partial^2u}{\partial x^2}$$

- D'Alambert solution, shown below, uses arbitrary functions F and G with coordinates ξ = x + ct and η = x – ct

$$u=F(\xi)+G(\eta)=F(x+ct)-G(x-ct)$$

- Proof of solution based on transforming derivatives

Derive D'Alambert Solution

- Transform equation from (x,t) to (ξ,η) using ξ = x + ct and η = x – ct

$$\frac{\partial}{\partial t}=\frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}=c\frac{\partial}{\partial\xi}-c\frac{\partial}{\partial\eta}$$

$$\frac{\partial}{\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=(1)\frac{\partial}{\partial\xi}+(1)\frac{\partial}{\partial\eta}$$

- Apply transforms to u, ∂u/∂t, and ∂u/∂x

$$\frac{\partial u}{\partial\xi}=\frac{\partial}{\partial\xi}[F(\xi)+G(\eta)]=F'(\xi)$$

$$\frac{\partial u}{\partial\eta}=\frac{\partial}{\partial\eta}[F(\xi)+G(\eta)]=G'(\eta)$$

Derive D'Alambert Solution II

- Continue transformations

$$\frac{\partial u}{\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial u}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial u}{\partial\eta}=(1)\frac{\partial[F(\xi)+G(\eta)]}{\partial\xi}+(1)\frac{\partial[F(\xi)+G(\eta)]}{\partial\eta}=F'(\xi)+G'(\eta)$$

$$\frac{\partial u}{\partial t}=\frac{\partial\xi}{\partial t}\frac{\partial u}{\partial\xi}+\frac{\partial\eta}{\partial t}\frac{\partial u}{\partial\eta}=c\frac{\partial[F(\xi)+G(\eta)]}{\partial\xi}-c\frac{\partial[F(\xi)+G(\eta)]}{\partial\eta}=c[F'(\xi)-G'(\eta)]$$

Derive D'Alambert Solution III

- Second derivatives satisfy wave equation

$$\frac{\partial^2u}{\partial t^2}=\frac{\partial}{\partial t}\frac{\partial u}{\partial t}=\frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}\frac{\partial u}{\partial t}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}\frac{\partial u}{\partial t}=c\frac{\partial}{\partial\xi}\frac{\partial u}{\partial t}-c\frac{\partial}{\partial\eta}\frac{\partial u}{\partial t}$$

$$=c\frac{\partial}{\partial\xi}[c(F'-G')]-c\frac{\partial}{\partial\eta}[c(F'-G')]=c^2(F''+G'')$$

$$\frac{\partial^2u}{\partial^2t}=$$

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial u}{\partial x}=\frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}\frac{\partial u}{\partial t}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}\frac{\partial u}{\partial t}=$$

$$(1)\frac{\partial}{\partial\xi}[(F'+G')]+(1)\frac{\partial}{\partial\eta}[(F'+G')]=(F''+G'')$$

$$c^2\frac{\partial^2u}{\partial x^2}$$

Solution with Initial Conditions

- Define $u(x,0) = f(x)$ and $\partial u/\partial t|_0 = g(x)$
- Solution, $u(x,t)$ uses $f(x \pm ct)$

$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\nu)d\nu$$

- Terms $f(x + ct)$ and $f(x - ct)$ satisfy wave equation since any function of these arguments satisfies the equation
- Solution gives $u(0,x) = f(x)$ as required

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Solution with Initial Conditions II

- Integral term satisfies wave equation
- Details of derivation in wave equation notes

$$\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} g(\nu)d\nu = c\left[g(x+ct) + g(x-ct)\right]$$

$$\frac{\partial^2}{\partial t^2}\int_{x-ct}^{x+ct} g(\nu)d\nu = c^2\left[g'(x+ct) - g'(x+ct)\right]$$

$$\frac{\partial^2}{\partial x^2}\int_{x-ct}^{x+ct} g(\nu)d\nu = g'(x+ct) + g'(x-ct) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{x-ct}^{x+ct} g(\nu)d\nu$$

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Solution with Initial Conditions III

- Verify initial condition $\partial u/\partial t|_0 = g(x)$

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\left[cf'(x+ct) + (-c)f'(x-ct)\right] + \frac{1}{2c}\left[cg(x+ct) + cg(x-ct)\right]$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \frac{1}{2}\left[cf'(x) + (-c)f'(x)\right] + \frac{1}{2c}\left[cg(x) + cg(x)\right] = g(x)$$

- Proposed solution satisfies wave equation and initial conditions

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Compare Solution Approaches

- Separation of variables solutions for $g(x) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

- Set $t = 0$ to get $f(x)$ initial condition

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- Compare to D'Alambert solution

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Compare Solutions II

- $f(x)$ from last chart gives $f(x \pm ct)$ below

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x+ct) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x+ct)}{L}\right) \qquad f(x-ct) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x-ct)}{L}\right)$$

- D'Alambert solution for $g(x) = 0$ is $u(x,t) = [f(x + ct) + f(x - ct)]/2$

$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\nu)d\nu$$

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Compare Solutions III

- D'Alambert solution: $[f(x + ct) + f(x - ct)]/2$

$$f(x+ct) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x+ct)}{L}\right) \qquad f(x-ct) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(x-ct)}{L}\right)$$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right]$$

- This matches separation-of-variables solution modified by trigonometric identities

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Why D'Alambert

- We see that the solution obtained by separation of variables agrees with the D'Alambert solution for one case
- The D'Alambert solution is more general
- It also provides a basis for propagation of wave shapes without damping
  - Look at meaning of f(x + ct) and f(x – ct)
    - If f(x) = a when x = b at t = 0 then at any point where x ± ct = b, f(x ± ct) = a

Meaning of D'Alambert Solution

- Consider case with g(x) = 0
- u(x,t) = [f(x + ct) + f(x – ct)]/2
- Initial condition is propagated into different spatial regions over time without change in shape
- Boundaries can affect solution
- Examine infinite region with simple f(x)
  - Triangular: f(x) = 1 + x (-1 ≤ x ≤ 0); f(x) = 1 – x (0 ≤ x ≤ 1) and f(x) = 0 otherwise

Meaning of Solution II

- Here is definition of f(z) for triangular wave from last chart (z = x ± ct)

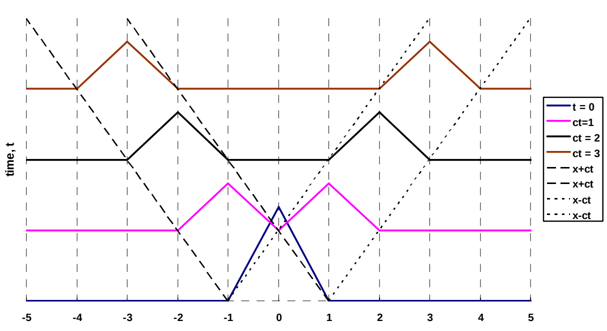
z	z < –1	–1≤ z < 0	0 ≤ z < 1	z ≥ 1
f(z)	0	1 + z	1 – z	0

- For any value of ct and x, we can find z = x ± ct and get the correct value of f(z) from this initial condition chart

Initial Wave Propagates

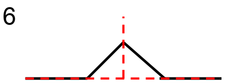
- For solution with g(x) = 0, u(x,t) = [f(x + ct) + f(x – ct)] /2
- Given x and ct we can compute f(x + ct) and f(x – ct) from table on previous chart
- Adding them together and dividing by 2 gives the solution for any u(x,t)
  - This is specific application of general idea to triangular initial condition

Wave Propagation



Triangular Initial Conditions

- Region 0 ≤ x ≤ L = 10 with g(x) = 0
- Triangular initial condition at center
  - f(x) = 0 for x ≤ 4 and x ≥ 6
  - f(x) = x – 4 for 4 ≤ x ≤ 5
  - f(x) = 6 – x for 5 ≤ x ≤ 6



$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{2}{10}\left[\int_0^4 (0)\sin\left(\frac{m\pi x}{10}\right)dx\right.$$

$$\left.+\int_4^5 (x-4)\sin\left(\frac{m\pi x}{10}\right)dx + \int_5^6 (6-x)\sin\left(\frac{m\pi x}{10}\right)dx + \int_6^{10} 0\right]$$

Triangular Initial Conditions II

$$B_m = \frac{2}{10}\left[\int_4^5 (x-4)\sin\left(\frac{m\pi x}{10}\right)dx + \int_5^6 (6-x)\sin\left(\frac{m\pi x}{10}\right)dx\right]$$

- Details of integration follow last chart of lecture presented in class

$$B_m = \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right]$$

- Can also use MATLAB to get B_m as shown on next chart

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B_m from MATLAB

- EDU>> syms m x I
- EDU>> (int((x - 4)*sin(m*pi*x/10),4,5) +int((6-x)*sin(m*pi*x/10),5,6))/5
- EDU>> I = simplify(ans)
- EDU>> pretty(I)

```
   2 sin(1/2 m pi) - sin(2/5 m pi) - sin(3/5 m pi)
20 -----------------------------------------------
                       2   2
                      m  pi
```

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What Happens at Boundaries

- Still have basic solution

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right]$$

- What if x ± ct is outside of range 0 ≤ x ≤ L?
- Solution in 0 ≤ x ≤ L will have components from periodic repetition of sine function just as in Fourier series

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Solution at Boundaries g(x) = 0

- Sine solution defined for limited region, but sine and cosine have periodic repetition for all values of their arguments

$$u(x,t) = \frac{f(x+ct) + f(x+ct)}{2} =$$

$$\frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right]$$

- f(x) = 0 for 0 ≤ x ≤ 0.4 and 0.6 ≤ x ≤ L = 1
- f(x) = 10x – 4 for 0.4 ≤ x ≤ 0.5
- f(x) = 6 – 10x for 0.5 ≤ x ≤ 0.6

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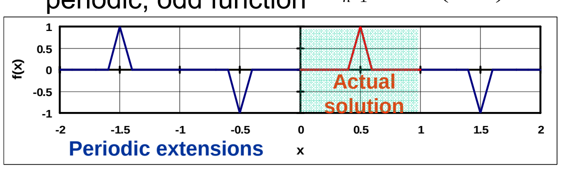
Solution is Fourier Series

- Wave equation solution is Fourier sine series which is periodic, odd function

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$



$$B_m = \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right]$$

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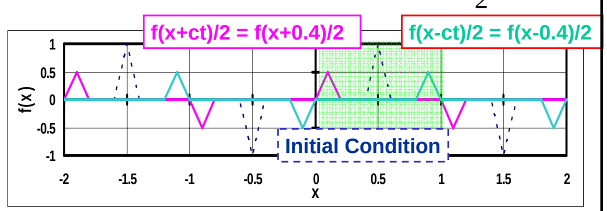
Time Evolution

- Look at evolution when ct = 0.4

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$



- For larger values of x ± ct, periodic extensions move into 0 ≤ x ≤ L = 1

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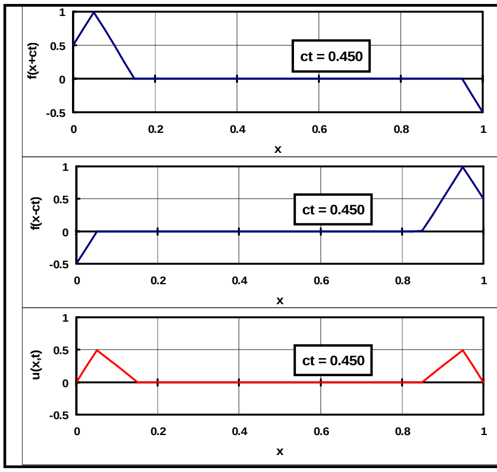

Phase behavior of sine function causes initial wave form to be reflected at bound-aries

Wave Equation Summary

- View spreadsheet showing wave travel for initial profiles
- Have separation-of-variables solution and D'Alambert solution

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\nu)d\nu$$

- D'Alambert solution shows how wave solution in x and t is composed of the initial profiles as traveling wave

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Overview of Characteristics

- Lines along which solution with discontinuities can propagate
- Main applications are for wave equation in which characteristics are real
- Has implications for understanding solutions and for numerical analysis
- Determine slope of characteristics by finding directions in which solution for equation is not unique

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Overview of Characteristics II

- See Hoffman for details of analysis for general second order PDE

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

- Analysis gives slope of characteristics

$$\frac{dy}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

- Characteristics slopes gives region of influence and domain of dependence

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Classification of PDEs

- The general second-order PDE in two variables is classified as follows
  - If $B^2 - 4AC < 0$ the PDE is called elliptic and has no real characteristic directions
  - If $B^2 - 4AC = 0$ the PDE is parabolic and has one repeated characteristic direction
  - If $B^2 - 4AC > 0$ the PDE is hyperbolic and has two real characteristic directions

$$\frac{dy}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

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Important Equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-D=0$$

Laplace/Poisson/Helmholtz equations ($B^2 - 4AC < 0$) are elliptic (no real characteristics)

Diffusion equation ($B^2 - 4AC = 0$) is parabolic (one characteristic)

$$\alpha\frac{\partial^2 u}{\partial x^2}+D=0$$

$$\frac{\partial^2 u}{\partial x^2}-c^2\frac{\partial^2 u}{\partial y^2}=0$$

Wave equation ($B^2 - 4AC > 0$) is hyperbolic (two real characteristics)

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Wave Equation Characteristics

- Compute characteristic directions

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Wave equation: A = 1; B = 0; C = –c^2; B^2 – 4AC = $c^2 > 0$

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial t \partial x} + C\frac{\partial^2 u}{\partial x^2} + D\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

$$\frac{dx}{dt} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-0 \pm \sqrt{0^2 - 4(1)(-c^2)}}{2(1)} = \pm c$$

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Wave Equation Characteristics II

- Compute characteristic directions with order of variables reversed

$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

Wave equation: A = c^2; B = 0; C = –1; B^2 – 4AC = $c^2 > 0$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

$$\frac{dt}{dx} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-0 \pm \sqrt{0^2 - 4(1)(-c^2)}}{2\left(-c^2\right)} = \pm\frac{1}{c}$$

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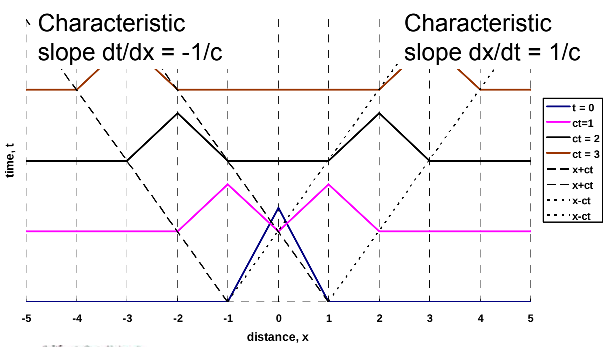
Wave Propagation



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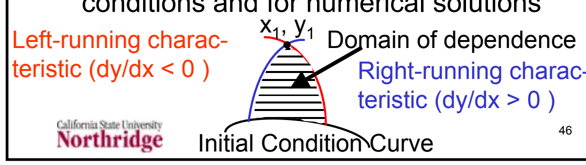
Behavior of Equation Types

- Domain of dependence for $u(x_1,y_1)$
  - The area (in x-y space) whose u values affect the value of $u(x_1,y_1)$
- Region of influence of $u(x_1,y_1)$
  - The area (in x-y space) whose u values are affected by the value of $u(x_1,y_1)$
- Importance for specifying boundary conditions and for numerical solutions



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Hyperbolic Equations

- Domain of dependence shown on previous chart
- Region of influence is region of characteristics leaving x_1, y_1
- Conditions outside domain of dependence should not affect solution
- Important point for numerical algorithms which can violate this principle for inappropriate choices of step sizes

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Elliptic PDEs

- Imaginary characteristics for elliptic equations like Laplace and Poisson's
- Entire solution region is both domain of dependence and region of influence
- This means that any change in any boundary condition can affect the solution at any point in the region
  - Effects may be small far from boundary, but will be present

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Parabolic PDEs

- Parabolic equations typically involve time and space as coordinates
- Consider region 0 ≤ x ≤ L and t > 0
  - Domain of dependence at any point x_1, t_1 is entire domain at previous times: 0 ≤ x ≤ L and 0 ≤ t < t_1
    - Any change in initial conditions or boundary conditions for t < t_1 will change solution here
  - Region of influence at x_1, t_1 is entire region for future times 0 ≤ x ≤ L and t > t_1

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Example Question

- You are solving the diffusion equation in the region 0 ≤ x ≤ L and t > 0 with an initial condition u(x,0) = f(x) and the following boundary conditions
  - t < 12 s: u(0,t) = u(L,t) = 0
  - t ≥ 12 s: u(0,t) = u(L,t) = a = 1
- If you have a solution to this problem for a = 1, how does the solution change for t < 12 s, if you set a = 2?

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March 9 Homework Problem

- Page 547, problem 12 – solve the wave equation for 0 ≤ x ≤ L = π and t ≥ 0 with boundary and initial conditions shown

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad c = 1 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0.01x & 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

$$u(0,t) = u(L,t) = 0 \qquad u(x,0) = 0$$

- Start with separation of variables result

$$u(x,t) = T(t)X(x) =$$
$$\left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

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March 9 Homework Problem II

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad c = 1 \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0.01x & 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

$$u(0,t) = u(L,t) = 0 \qquad u(x,0) = 0$$

$$u(x,t) = T(t)X(x) =$$
$$\left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

- For u(0,t) = 0 we must have D = 0
- For u(L,t) = 0 we must have λL = nπ (n an integer)
- Need to treat initial conditions

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March 9 Homework Problem III

$$u(x,t) = \left[A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)\right]\sin(\lambda_n x)$$

- From previous separation of variables solution we have

$$A_m = \frac{2}{m\pi c}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

- In this problem f(x) = u(x,0) = 0 so we have all B_m = 0

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March 9 Homework Problem IV

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n ct)\sin(\lambda_n x)$$

- Use function given for g(x) to get A_m

$$A_m = \frac{2}{m\pi c}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$A_m = \frac{2}{m\pi c}\int_0^{\frac{\pi}{2}} 0.01x\sin\left(\frac{m\pi x}{L}\right)dx + \frac{2}{m\pi c}\int_{\frac{\pi}{2}}^{\pi} 0.01(\pi - x)\sin\left(\frac{m\pi x}{L}\right)dx$$

- Set L = π and c = 1

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Conclusions

- Wave equation can be solved by conventional separation of variables
- D'Alambert solution is special approach for wave equation
  - Consistent with separation of variables solution, but more general
  - Shows propagation in terms like f(x + ct) and f(x – ct)

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Notes

- The remaining charts show the integration of the expansions coefficients, B_m, for the triangular initial condition
- The first midterm is scheduled for Thursday, March 4 will cover material up to and including the March 2 homework

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Get Integral Terms

- Constant terms in B_m integral

$$\frac{2}{10}\left[\int_4^5 (-4)\sin\left(\frac{m\pi x}{10}\right)dx + \int_5^6 (6)\sin\left(\frac{m\pi x}{10}\right)dx\right] =$$

$$\frac{2}{10}\frac{10}{m\pi}\left\{\left[4\cos\left(\frac{m\pi x}{10}\right)\right]_4^5 - \left[6\cos\left(\frac{m\pi x}{10}\right)\right]_5^6\right\} =$$

$$\frac{2}{m\pi}\left[4\cos\left(\frac{m\pi 5}{10}\right) - 4\cos\left(\frac{m\pi 4}{10}\right) - 6\cos\left(\frac{m\pi 6}{10}\right) + 6\cos\left(\frac{m\pi 5}{10}\right)\right]$$

$$= \frac{20}{m\pi}\cos\left(\frac{m\pi}{2}\right) - \frac{8}{m\pi}\cos\left(\frac{2m\pi}{5}\right) - \frac{12}{m\pi}\cos\left(\frac{3m\pi}{5}\right)$$

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Use $\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$

- x terms in B_m integral

$$I_1 = \frac{2}{10}\int_4^5 x\sin\left(\frac{m\pi x}{10}\right)dx = \frac{2}{10}\left(\frac{10}{m\pi}\right)^2\left[\sin\left(\frac{m\pi x}{10}\right) - \frac{m\pi x}{10}\cos\left(\frac{m\pi x}{10}\right)\right]_4^5$$

$$= \frac{2}{10}\left(\frac{10}{m\pi}\right)^2\left[\sin\left(\frac{m\pi 5}{10}\right) - \sin\left(\frac{m\pi 4}{10}\right) - \frac{m\pi 5}{10}\cos\left(\frac{m\pi 5}{10}\right) + \frac{m\pi 4}{10}\cos\left(\frac{m\pi 4}{10}\right)\right]$$

$$I_2 = \frac{2}{10}\int_5^6 x\sin\left(\frac{m\pi x}{10}\right)dx = \frac{2}{10}\left(\frac{10}{m\pi}\right)^2\left[\sin\left(\frac{m\pi x}{10}\right) - \frac{m\pi x}{10}\cos\left(\frac{m\pi x}{10}\right)\right]_5^6$$

$$= \frac{2}{10}\left(\frac{10}{m\pi}\right)^2\left[\sin\left(\frac{m\pi 6}{10}\right) - \sin\left(\frac{m\pi 5}{10}\right) - \frac{m\pi 6}{10}\cos\left(\frac{m\pi 6}{10}\right) + \frac{m\pi 5}{10}\cos\left(\frac{m\pi 5}{10}\right)\right]$$

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x Terms Continued

$$\int_4^5 x\sin\left(\frac{m\pi x}{10}\right)dx - \int_5^6 x\sin\left(\frac{m\pi x}{10}\right)dx = I_1 - I_2$$

$$= \frac{20}{m^2\pi^2}\left[\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \frac{m\pi}{2}\cos\left(\frac{m\pi}{2}\right) + \frac{2m\pi}{5}\cos\left(\frac{2m\pi}{5}\right)\right]$$

$$- \frac{20}{m^2\pi^2}\left[\sin\left(\frac{3m\pi}{5}\right) - \sin\left(\frac{m\pi}{2}\right) - \frac{3m\pi}{5}\cos\left(\frac{3m\pi}{5}\right) + \frac{m\pi}{2}\cos\left(\frac{m\pi}{2}\right)\right]$$

$$= \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right.$$

$$\left. - m\pi\cos\left(\frac{m\pi}{2}\right) + \frac{2m\pi}{5}\cos\left(\frac{2m\pi}{5}\right) + \frac{3m\pi}{5}\cos\left(\frac{3m\pi}{5}\right)\right]$$

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Result for B_m

$$B_m = \frac{2}{10}\left[\int_0^4 (0)\sin\left(\frac{m\pi x}{10}\right)dx + \int_4^5 (x-4)\sin\left(\frac{m\pi x}{10}\right)dx + \int_5^6 (6-x)\sin\left(\frac{m\pi x}{10}\right)dx + 0\right]$$

$$= \frac{20}{m\pi}\cos\left(\frac{m\pi}{2}\right) - \frac{8}{m\pi}\cos\left(\frac{2m\pi}{5}\right) - \frac{12}{m\pi}\cos\left(\frac{3m\pi}{5}\right)$$

$$+ \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right.$$

$$\left. - m\pi\cos\left(\frac{m\pi}{2}\right) + \frac{2m\pi}{5}\cos\left(\frac{2m\pi}{5}\right) + \frac{3m\pi}{5}\cos\left(\frac{3m\pi}{5}\right)\right]$$

These terms all cancel (note factor of $20/m^2\pi^2$ multiplying last row)

$$B_m = \frac{20}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \sin\left(\frac{2m\pi}{5}\right) - \sin\left(\frac{3m\pi}{5}\right)\right]$$

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