Laplace Equation Conclusion and The Wave Equation

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Mechanical Engineering 501B

Seminar in Engineering Analysis

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Overview

- Review material to date
  - – General approach for solving PDEs
- Other ideas about Laplace's Equation
- Derivation and physical meaning of wave equation
- Solution of the wave equation by separation of variables
- Introduction to the D'Alambert solution of the wave equation
- Midterm Exam, Wednesday, March 4

Solving PDEs

- Perform necessary operations if boundaries not homogenous
  - – Diffusion: define u(**x**,t) = v(**x**,t) + w(**x**)
    - v satisfies diffusion equation with zero boundary conditions; w satisfies boundary
  - – Laplace equation use superposition
    - Solution is sum of two or more solutions each of which has only one nonzero boundary
- Get equation in appropriate coordinate system (rectangular or cylindrical)

Solving PDEs II

- Get appropriate separation of variables solution
  - – Want eigenfunctions to express initial or boundary condition
- Use homogenous boundary conditions to determine constants in solutions and eigenvalues
- General, sum of all eigenfunctions, used to fit initial or boundary condition

Midterm Exam

- Wednesday, March 4
- Covers material on diffusion and Laplace equations
- Includes material up to and including tonight's lecture and homework for Monday, March 2
- Open book and notes, including homework solutions

Review Spherical Laplace

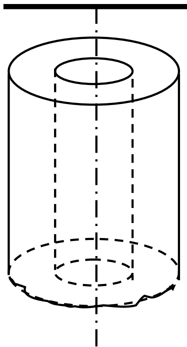
- Laplace's equation in a sphere has solutions in Legendre polynomials, $P_n(x)$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)=0$$

$$u(r,\varphi)=\sum_{n=0}^{\infty}P_n(\cos\phi)\left(A_n r^n+B_n r^{-1-n}\right)$$

$$P_n(x)=\sum_{m=0}^{\mathrm{int}(n/2)}(-1)^n\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

Review Hollow Cylinder



- Consider various boundary conditions
- Nonzero conditions on upper or lower surface only gives Bessel eignefunctions
- Nonzero conditions on inner or outer surface gives sine or cosine eigenfunctions

Review Hollow Cylinder II

- Laplace’s Equation in two-dimensional cylindrical region $0 \le z \le L$ and $R_i \le r \le R_o$
  - $u(r,0) = u(r,L) = u(R_i,z) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \quad \boxed{u(R_o,z)=u_R(z)}$$

$$u(r,z)=\sum_{m=1}^{\infty}C_m\sin(\lambda_m z)\left[I_0(\lambda_m r)-\frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m r)\right] \quad \lambda_m=\frac{m\pi}{L}$$

$$C_m=\frac{2K_0(\lambda_m R_i)}{[I_0(\lambda_m R_0)K_0(\lambda_m R_i)-I_0(\lambda_m R_i)K_0(\lambda_m R_0)]}\int_0^L \sin(\lambda_m z)u_R(z)dz$$

Review Hollow Cylinder III

- Laplace’s Equation in two-dimensional cylindrical region $0 \le z \le L$ and $R_i \le r \le R_o$
  - $u(r,0) = u(R_i,z) = u(R_o,z) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \quad \boxed{u(r,L)=u_N(r)}$$

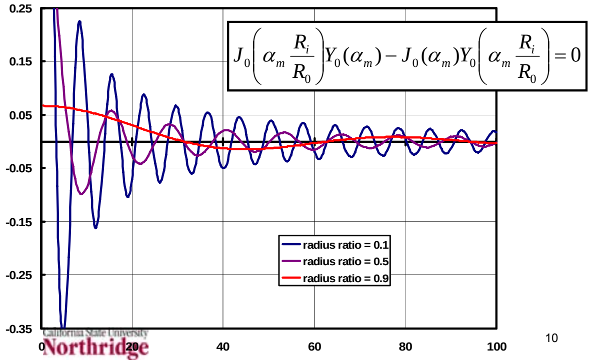
- Eigenvalues, $\lambda_m = \alpha_m/R_0$ and eigenfunction

$$J_0\left(\alpha_m\frac{R_i}{R_0}\right)Y_0(\alpha_m)-J_0(\alpha_m)Y_0\left(\alpha_m\frac{R_i}{R_0}\right)=0$$

$$u(r,z)=\sum_{m=1}^{\infty}C_m\sinh(\lambda_m z)\boxed{[Y_0(\lambda_m R_0)J_0(\lambda_m r)-J_0(\lambda_m R_0)Y_0(\lambda_m r)]}$$

Eigenfunction, $P_0(\lambda_m r)$

Review Eigenvalues = f(R_i/R_0)



$$J_0\left(\alpha_m\frac{R_i}{R_0}\right)Y_0(\alpha_m)-J_0(\alpha_m)Y_0\left(\alpha_m\frac{R_i}{R_0}\right)=0$$

Review Hollow Cylinder IV

- Eigenfunction expansion in $P_0(\lambda_m r) = Y_0(\lambda_m R_0)J_0(\lambda_m r) - J_0(\lambda_m R_0) \cdot Y_0(\lambda_m r)$

$$C_m=\frac{[\pi\lambda_m J_0(\lambda_m R_i)]^2\int_{R_i}^{R_0} r u_N(r)P_0(\lambda_m r)dr}{\sinh(\lambda_m L)2[J_0^2(\lambda_m R_i)-J_0^2(\lambda_m R_0)]}$$

- Solution for $u_N(r) = U$, a constant

$$u(r,z)=U\pi\sum_{m=1}^{\infty}\frac{\sinh(\lambda_m z)}{\sinh(\lambda_m L)}\frac{J_0(\lambda_m R_i)P_0(\lambda_m r)}{J_0(\lambda_m R_i)+J_0(\lambda_m R_0)}$$

Review Conclusions

- Approach to solving Laplace equation is similar to that of diffusion equation
  - Main difference is that second dimension (y or r) in Laplace equation gives closed boundary instead of open boundary in time
  - Use separation of variables
  - Have eigenfunction solution (sine/cosine, Bessel or other) in one dimension
  - Use eigenfunction expansion to fit condition at one boundary

Review Conclusions II

- Use superposition to solve Laplace equation with more than one nonzero boundary
- Additional cylindrical geometry considerations
  - Complex Bessel functions when radial boundary is not eigenfunction solution
  - Must include both Y_0 and J_0 when radial coordinate does not start at zero (must have zero boundary at inner radius)

Vector Calculus

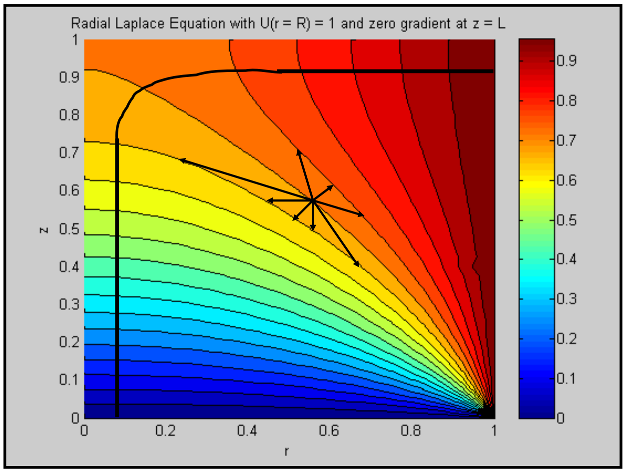
- Important results for Laplace equation
- See notes on vector calculus or chapters nine and ten in Kreyszig for background details not given here
- Results are independent of coordinate system, but Cartesian used for examples
- Introduce gradient and divergence which are vector/scalar functions

Gradients

- Gradient is a vector in written here in Cartesian space where we have f(x,y,z)
- Definition of gradient $grad\ f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$
- Del operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
- grad f is magnitude and direction of maximum gradient, df/ds
- Grad f is perpendicular to line of constant f

Physical Gradients

- Gradients of Laplace equation solutions often proportional to flux terms
  - Heat flux and temperature gradient
  - Diffusion flux and mass fraction gradient
  - Velocity and velocity potential in ideal flow
  - Current and electrostatic potential
- If we have a plot of constant potential the lines perpendicular to the potential are flux lines



Divergence

- Divergence converts vector, $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, into a scalar written as div **v**
- Definition of divergence $div\ \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
- Del operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
- Gauss divergence theorem (**n** is vector normal to surface, pointing outward)

$$\underset{\substack{Enclosed\\Volume}}{\iiint} div\ \mathbf{v}dV = \underset{Surface}{\iint} \mathbf{v}\cdot\mathbf{n}dA$$

$$\iiint\limits_{\substack{Enclosed\\Volume}} div\,\mathbf{q}dV = \iint\limits_{Surface}\mathbf{q}\cdot\mathbf{n}dA$$

- Example of heat flux vector, **q**, (W/m^2)
- **q·n** is component of **q**, normal to surface, dA, flowing outward
- Integrand in surface integral, **q·n**dA is heat flow (watts) flowing out through infinitesimal area, dA
- Surface integral gives total heat flow through surface in outward direction

Relation to Laplace Equation

- The vector, **v**, may be gradient of a scalar, representing a flux: **v** = -k grad u

$$\iiint\limits_{\substack{Enclosed\\Volume}} div\;\mathbf{v}dV = \iiint\limits_{\substack{Enclosed\\Volume}} div\left(-k\;grad\;u\right)dV = \iint\limits_{Surface}\mathbf{v}\cdot\mathbf{n}dA$$

- For constant k

$$\iiint\limits_{\substack{Enclosed\\Volume}} div\left(grad\;u\right)dV = \iiint\limits_{\substack{Enclosed\\Volume}} \nabla^2 u dV = -\frac{1}{k}\iint\limits_{Surface}\mathbf{v}\cdot\mathbf{n}dA$$

Interpretation of $\nabla^2 u = 0$

- When **v** = -k grad u is a flux that is the gradient of a scalar, Laplace's equation for u says that the net inflow of **v** is zero

$$\iiint\limits_{\substack{Enclosed\\Volume}} \nabla^2 u dV = -\frac{1}{k}\iint\limits_{Surface}\mathbf{v}\cdot\mathbf{n}dA = 0$$

- Example of this result shown last week
- Result applies to any problem in any geometry with Laplace's equation

Complex Variable Basics

- Complex analysis gives insights to Laplace Equation in two dimensions
- Functions of complex variable, z = x + iy: f(z) = u(x,y) + iv(x,y), for example
  - f(z) = z^2 = (x + iy)^2 = x^2 + 2ixy – y^2
  - f(z) = u = x^2 + y^2 and v = 2xy
- What is df/dz? Is it unique?
  - Cauchy-Riemann conditions

$$If\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\quad and\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\quad then\quad \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy-Riemann Example

- f(z) = z^2 = x^2 – y^2 + 2ixy so u(x,y) = x^2 – y^2 and v(x,y) = 2xy

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x \quad and \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} = 2y$$

- Satisfies Cauchy-Riemann conditions

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2z$$

$$\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 2x + i2y = 2z$$

Connection to Laplace Equation

- Take ∂/∂x of first Cauchy-Riemann condition and ∂/∂y of second one and add the results to get Laplace equation

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right] \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right] \Rightarrow \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Connection to Laplace Equation

- Take ∂/∂y of first Cauchy-Riemann condition and ∂/∂y of second one and subtract for another Laplace equation

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\right] \Rightarrow \frac{\partial^2 u}{\partial y \partial x}=\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\right] \Rightarrow \frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Important Result

- If there is a function u(x,y) that satisfies a Laplace equation in two dimensions, there is an associated function v(x,y) that also satisfies Laplace's equation
- The lines of u(x,y) and v(x,y) are mutually perpendicular
- Typically if u is a potential (e.g, temperature, v is a corresponding flux)

Orthogonal Solutions

- Show that the two solutions u and v are orthogonal at all points
- Consider the gradient of each function

$$\nabla u=\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j} \qquad and \qquad \nabla v=\frac{\partial v}{\partial x}\mathbf{i}+\frac{\partial v}{\partial y}\mathbf{j}$$

- Take the dot product of the gradients

$$\nabla u\cdot\nabla v=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\mathbf{i}\cdot\mathbf{i}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\mathbf{j}\cdot\mathbf{j}+\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right]\mathbf{i}\cdot\mathbf{j}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

Additional Results

- Treat Laplace equation solutions as complex variable F(z) = u(x,y) + i v(x,y)
- Cauchy theorem for complex integration shows Laplace equation solutions
  - Solutions called harmonic functions
  - Have maximum and minimum on boundary
  - If boundary is a constant at all points then solution is the same constant in region
  - Dirichlet problem has unique solution
- Kreyszig section 18.6 has proofs

Neumann Problem is not Unique

- Consider the following problem

$$\nabla^2 u=0 \text{ in a region with } \frac{\partial u}{\partial n} \text{ specified on its boundaries}$$

- If u_1 satisfies the differential equation and boundary conditions,
  - Any other solution = u_1 plus a constant will also satisfy the problem conditions
  - Conclusion: at least part of the boundary must have Dirichlet or third kind of boundary condition

Conclusions

- Approach to solving Laplace equation is similar to that of diffusion equation
  - Main difference is that second dimension (y) in Laplace equation gives closed boundary instead of open boundary in time
  - Use separation of variables
  - Have eigenfunction solution (sine/cosine, Bessel or other) in one dimension
  - Use eigenfunction expansion to fit condition at one boundary

Conclusions II

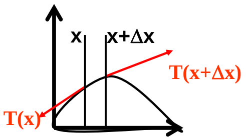
- Use superposition to solve Laplace equation with more than one nonzero boundary
- Additional cylindrical geometry considerations
  - Complex Bessel functions when radial boundary is not eigenfunction solution
  - Must include both Y_0 and J_0 when radial coordinate does not start at zero (must have zero boundary at inner radius)

Conclusions III

- Results about Laplace's equation from vector analysis and complex variables
- When the gradient of the dependent variable, such as T, in Laplace's equation represents a flux, Laplace's equation says the net outflow is zero
- The maximum and minimum values of a solution to Laplace's equation occur on the boundary so a constant boundary means a constant solution

Wave Equation Derivation

- Consider a string fixed at x = 0 and L
- String tension, **T**, tangential to string
- ρ is mass per unit length of string
- Section between x and x + Δx
- Net y force = mass times acceleration



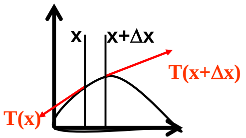
$$T(x+\Delta x)\sin(\theta+\Delta\theta) - T(x)\sin\theta = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$$

Wave Equation Derivation

- Previous equation for vertical component, v, in limit as Δx approaches zero

$$\frac{T(x+\Delta x)\sin(\theta+\Delta\theta) - T(x)\sin\theta}{\Delta x} = \frac{\partial v}{\partial x} = \rho\frac{\partial^2 y}{\partial t^2}$$

- Horizontal component, h, is related to v as v = h tan θ = h dy/dx; h is constant



$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}h\frac{\partial y}{\partial x} = h\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad c^2 = \frac{h}{\rho}$$

Wave Equation

- Wave phenomena: u(x,t) is wave amplitude varying with space, x, and time, t

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

- c is wave speed
- Can solve by usual separation of variables technique
- Also have D'Alambert solution with arbitrary functions F and G with coordinates ξ = x + ct and η = x – ct

Separation of Variables

- Assume u(x,t) = X(x)T(t)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2[X(x)T(t)]}{\partial t^2} = X(x)\frac{\partial^2 T(t)}{\partial t^2} =$$

$$c^2\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2[X(x)T(t)]}{\partial x^2} = c^2 T(t)\frac{\partial^2 X(x)}{\partial x^2}$$

- Divide by $c^2X(x)T(t)$

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$$

Result is function of t equal to function of x

Separation of Variables Works

- Assumed solution, u(x,t) = X(x)T(t), gives a function of x equal a function of t
- Since x and t are independent, both sides must equal a constant for this to be true

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2$$

- Choose constant as –λ² to simplify later solution of resulting ordinary differential equations (ODE)

Solve ODEs to Get u(x,t)

- Have simple ODEs with known general solutions

$$\frac{d^2T(t)}{dt^2} + \lambda^2 c^2 T(t) = 0 \quad T(t) = A\sin(\lambda ct) + B\cos(\lambda ct)$$

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \quad X(x) = C\sin(\lambda x) + D\cos(\lambda x)$$

$$u(x,t) = T(t)X(x) =$$
$$\left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x) + D\cos(\lambda x)\right]$$

Boundary Conditions

- Look at case where u(0,t) = u(L,t) = 0
- In this case, X(x) is the solution to a Sturm-Liouville problem
- X(x) = Csin(λx) + Dcos(λx)
- X(0) = 0 = Csin(0) + Dcos(0) = D = 0
- X(L) = 0 = Csin(λL)
- Must have λL = nπ (n an integer)
- Eigenfunction is sin(nπx/L)

General Solution

- With C = 0, and λ_n = nπ/L, our solution is

$$u(x,t) = \left[A\sin(\lambda ct) + B\cos(\lambda ct)\right]\left[C\sin(\lambda x)\right] =$$
$$\left[A_n\sin(\lambda_n ct) + B_n\cos(\lambda_n ct)\right]\left[\sin(\lambda_n x)\right]$$

- General solution sum of all eigenfunctions

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\sin(\lambda_n ct) + B_n\cos(\lambda_n ct)\right]\sin(\lambda_n x)$$

  - Use eigenfunction expansions to get initial conditions

Initial Conditions

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\sin(\lambda_n ct) + B_n\cos(\lambda_n ct)\right]\sin(\lambda_n x)$$

- Specify u(x,0) = f(x) at t = 0 and $\partial u/\partial t|_{t=0}$ = g(x) at t = 0

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} c\lambda_n A_n \sin\left(\frac{n\pi x}{L}\right)$$

Orthogonal Eigenfunctions

- Multiply by sin(mπx/L) and integrate

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} B_n \int_0^{x_{max}} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx$$

- Uses orthogonality for eigenfunctions

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \delta_{mn}\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{L\delta_{mn}}{2}$$

Similar Result for A_n

- Multiply by sin(mπx/L) and integrate

$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} c\lambda_n A_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \int_0^{x_{\max}} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{cm\pi}{L} A_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx$$

- Factor multiplying A_m is

$$\frac{cm\pi}{L}\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{cm\pi}{L}\frac{L}{2} = \frac{cm\pi}{2}$$

Summary: A_m and B_m

$$B_m = \frac{\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$A_m = \frac{\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\frac{cm\pi}{L}\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx} = \frac{2}{m\pi c}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

- If g(x) = 0, all A_m are zero

General Solution

- Substitute A_m and B_m from equations just found and substitute into previous solution

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

- Examine case where g(x) = 0 so A_n = 0

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

General Solution for g(x) = 0

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

- From trig identities for sin(x ± y)
  - sin(x + y) = sin x cos y + sin y cos x
  - sin(x – y) = sin x cos y – sin y cos x
  - sin(x + y) + sin(x – y) = 2 sin x cos y

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\sin\left(\frac{n\pi(x+ct)}{L}\right) + \sin\left(\frac{n\pi(x-ct)}{L}\right)\right]$$

D'Alambert Solution

- Wave phenomena: u(x,t) is wave amplitude varying with space, x, and time, t

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

- c is wave speed
- D'Alambert solution, shown below, uses arbitrary functions F and G with coordinates ξ = x + ct and η = x – ct

$$u = F(\xi) + G(\eta) = F(x+ct) - G(x-ct)$$

- Proof of solution based on transforming derivatives

D'Alambert Solution

- Transform equation from (x,t) to (ξ,η) using ξ = x + ct and η = x – ct

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = (1)\frac{\partial}{\partial \xi} + (1)\frac{\partial}{\partial \eta}$$

- Apply transforms to u, ∂u/∂t, and ∂u/∂x

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left[F(\xi) + G(\eta)\right] = F'(\xi)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}\left[F(\xi) + G(\eta)\right] = G'(\eta)$$

D'Alambert Solution II

- Continue transformations

$$\frac{\partial u}{\partial x}=\frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi}+\frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta}=(1)\frac{\partial[F(\xi)+G(\eta)]}{\partial \xi}$$

$$+(1)\frac{\partial[F(\xi)+G(\eta)]}{\partial \eta}=F'(\xi)+G'(\eta)$$

$$\frac{\partial u}{\partial t}=\frac{\partial \xi}{\partial t}\frac{\partial u}{\partial \xi}+\frac{\partial \eta}{\partial t}\frac{\partial u}{\partial \eta}=c\frac{\partial[F(\xi)+G(\eta)]}{\partial \xi}$$

$$-c\frac{\partial[F(\xi)+G(\eta)]}{\partial \eta}=c\left[F'(\xi)-G'(\eta)\right]$$

D'Alambert Solution III

- Second derivatives satisfy wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\frac{\partial u}{\partial t}=\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\frac{\partial u}{\partial t}+\frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\frac{\partial u}{\partial t}=c\frac{\partial}{\partial \xi}\frac{\partial u}{\partial t}-c\frac{\partial}{\partial \eta}\frac{\partial u}{\partial t}$$

$$=c\frac{\partial}{\partial \xi}\left[c(F'-G')\right]-c\frac{\partial}{\partial \eta}\left[c(F'-G')\right]=c^2(F''+G'')$$

$$\frac{\partial^2 u}{\partial^2 t}=$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial u}{\partial x}=\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\frac{\partial u}{\partial t}+\frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\frac{\partial u}{\partial t}=$$

$$(1)\frac{\partial}{\partial \xi}\left[(F'+G')\right]+(1)\frac{\partial}{\partial \eta}\left[(F'+G')\right]=(F''+G'')$$

$$c^2\frac{\partial^2 u}{\partial x^2}$$