Equations in Spheres, Vector Calculus and Complex Variables

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Seminar in Engineering Analysis

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Overview

- Review last two classes
  - Solutions of Laplace's equation in cylindrical coordinates
- Spherical coordinate systems
- Diffusion equation in a sphere
- Laplace equation in a sphere
- Legendre polynomials as orthogonal eigenfunctions
- Results from complex analysis and vector calculus for Laplace's equation

Review Cylindrical Solutions

- Laplace's Equation in two-dimensional cylindrical region 0 ≤ z ≤ L and 0 ≤ r ≤ R
  - u(r,0) = 0 and u(0,z) is finite

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \quad u(R,z)=u_R(z) \quad \left.\frac{\partial u}{\partial z}\right|_{z=L}=0$$

$$u(r,z)=\sum_{m=0}^{\infty}C_m \sin(\lambda_m z)I_0(\lambda_m r) \quad \lambda_m=(2m+1)\frac{\pi}{2L}$$

$$C_m=\frac{2}{I_0(\lambda_m R)L}\int_0^L \sin(\lambda_m z)u_R(z)dz$$

Review Cylindrical Solutions II

- Laplace's Equation in two-dimensional cylindrical region 0 ≤ z ≤ L and 0 ≤ r ≤ R
  - u(r,0) = 0 and u(0,z) is finite

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \quad u(R,z)=u_R(z) \quad u(r,L)=0$$

$$u(r,z)=\sum_{m=0}^{\infty}C_m \sin(\lambda_m z)I_0(\lambda_m r) \quad \lambda_m=\frac{m\pi}{L}$$

$$C_m=\frac{2}{I_0(\lambda_m R)L}\int_0^L \sin(\lambda_m z)u_R(z)dz$$

Modified Bessel Functions

- $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $K_\nu(x) = i^{-\nu}Y_\nu(ix)$, $i^2 = -1$
- Satisfy modified differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+\nu^2\right)y=0$$

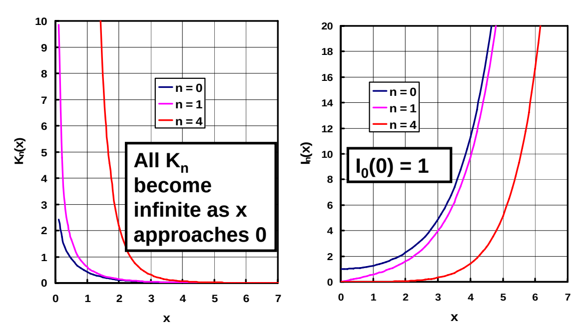
Since $J_\nu \sim x^\nu$, I_ν and K_ν are real

- Equation above transforms to

$$\frac{d}{dz}\left[z\frac{dy}{dz}\right]+\left(\frac{\nu^2}{z}-\lambda^2 z\right)y=0$$

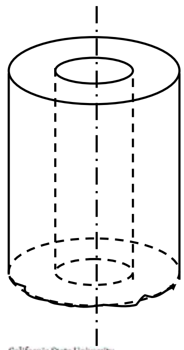
- Solution is $z = AI_\nu(\lambda x) + BK_\nu(\lambda x)$

Review Modified Bessel Functions



$I_\nu(x) = i^{-\nu}J_\nu(ix)$, $K_\nu(x) = i^{-\nu}Y_\nu(ix)$

Review Hollow Cylinder



- Consider various boundary conditions
- Nonzero conditions on upper or lower surface only gives Bessel eignefunctions
- Nonzero conditions on inner or outer surface gives sine or cosine eigenfunctions

Review Hollow Cylinder II

- Laplace's Equation in two-dimensional cylindrical region $0 \le z \le L$ and $R_i \le r \le R_o$
  - $u(r,0) = u(r,L) = u(R_i,z) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad u(R_o, z) = u_R(z)$$

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sin(\lambda_m z)\left[I_0(\lambda_m r) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)} K_0(\lambda_m r)\right] \qquad \lambda_m = \frac{m\pi}{L}$$

$$C_m = \frac{2K_0(\lambda_m R_i)}{\left[I_0(\lambda_m R_0)K_0(\lambda_m R_i) - I_0(\lambda_m R_i)K_0(\lambda_m R_0)\right]} \int_0^L \sin(\lambda_m z) u_R(z)\,dz$$

Review Hollow Cylinder III

- Laplace's Equation in two-dimensional cylindrical region $0 \le z \le L$ and $R_i \le r \le R_o$
  - $u(r,0) = u(R_i,z) = u(R_o,z) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad u(r, L) = u_N(r)$$

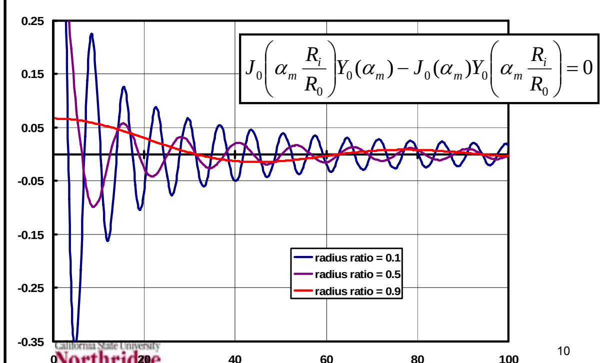
- Eigenvalues, $\lambda_m = \alpha_m / R_0$ and eigenfunction

$$J_0\left(\alpha_m \frac{R_i}{R_0}\right) Y_0(\alpha_m) - J_0(\alpha_m) Y_0\left(\alpha_m \frac{R_i}{R_0}\right) = 0$$

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m z)\left[Y_0(\lambda_m R_0) J_0(\lambda_m r) - J_0(\lambda_m R_0) Y_0(\lambda_m r)\right]$$

Eigenfunction, $P_0(\lambda_m r)$

Review Eigenvalues = $f(R_i/R_0)$



Review Hollow Cylinder IV

- Eigenfunction expansion in $P_0(\lambda_m r) = Y_0(\lambda_m R_0) J_0(\lambda_m r) - J_0(\lambda_m R_0) \cdot Y_0(\lambda_m r)$

$$C_m = \frac{\left[\pi \lambda_m J_0(\lambda_m R_i)\right]^2 \int_{R_i}^{R_0} r u_N(r) P_0(\lambda_m r)\,dr}{\sinh(\lambda_m L) 2\left[J_0^2(\lambda_m R_i) - J_0^2(\lambda_m R_0)\right]}$$

- Solution for $u_N(r) = U$, a constant

$$u(r,z) = U\pi \sum_{m=1}^{\infty} \frac{\sinh(\lambda_m z)}{\sinh(\lambda_m L)} \frac{J_0(\lambda_m R_i) P_0(\lambda_m r)}{J_0(\lambda_m R_i) + J_0(\lambda_m R_0)}$$

Review Conclusions

- Approach to solving Laplace equation is similar to that of diffusion equation
  - Main difference is that second dimension (y or r) in Laplace equation gives closed boundary instead of open boundary in time
  - Use separation of variables
  - Have eigenfunction solution (sine/cosine, Bessel or other) in one dimension
  - Use eigenfunction expansion to fit condition at one boundary

Review Conclusions II

- Use superposition to solve Laplace equation with more than one nonzero boundary
- Additional cylindrical geometry considerations
  - Complex Bessel functions when radial boundary is not eigenfunction solution
  - Must include both Y_0 and J_0 when radial coordinate does not start at zero (must have zero boundary at inner radius)

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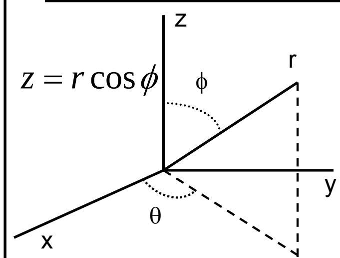
Spherical Coordinates



$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$

- Coordinate system used in Kreyszig
  - See p A72
  - ϕ is called the polar angle
  - θ is same in cylindrical system
- Some works reverse θ and ϕ

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Spherical Geometry

- Look at diffusion equation in a sphere

$$\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)$$

- Spherical symmetry has r derivatives only $\quad \frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$
- Initial condition, u(r,0) = f(r)
- Boundary conditions u(R,t) = u_R and u(0,t) is finite

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Spherical Geometry II

- Write u(r,t) = v(r,t) + u_R and use separation of variables: v(r,t) = T(t)P(r)

$$\frac{1}{\alpha T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{r^2 P(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial P(r)}{\partial r}\right) = -\lambda^2$$

$$\frac{dT(t)}{dt} + \lambda^2\alpha T(t) = 0 \qquad T(t) = Ae^{-\alpha t}$$

$$\frac{d}{dr}\left(r^2\frac{dP(r)}{dr}\right) + \lambda^2 r^2 P(r) = 0$$

- Sturm-Liouville problem with weight function p(r) = r^2

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Spherical Geometry III

- Define W(r) such that for P(r) = W(r) / r and radial equation becomes

$$\frac{d^2W(r)}{dx^2} + \lambda^2 W(r) = 0 \qquad W = A\sin\lambda r + B\cos\lambda r$$

$$P(r) = \frac{W(r)}{r} = A\frac{\sin\lambda r}{r} + B\frac{\cos\lambda r}{r}$$

- Finite solution at r = 0 gives B = 0 and v = 0 at R = r gives λ_n = nπr/R

$$u(r,t) = v(r,t) + u_R = \sum_{n=1}^{\infty}\frac{C_n e^{-\lambda_n^2\alpha t}\sin(\lambda_n r)}{r} + u_R$$

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Spherical Geometry IV

- Use eigenfunction expansion for initial conditions to evaluate constants, C_n

$$u(r,0) = f(r) = v(r,0) + u_R = \sum_{n=1}^{\infty}\frac{C_n\sin(\lambda_n r)}{r} + u_R$$

$$C_m = \frac{\int_0^R \left(f(r) - u_R\right) r^2 \frac{\sin\left(\frac{m\pi r}{R}\right)}{r}\,dr}{\int_0^R r^2\frac{\sin^2\left(\frac{m\pi r}{R}\right)}{r^2}\,dr}$$

- Need weight function, p(r) = r^2

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Spherical Geometry V

- Look at Laplace equation for a sphere

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)=0$$

- Consider no variation in θ coordinate

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)=0$$

- Boundary conditions $u(R,\phi) = f(\phi)$
- u is finite at r = 0 and as $r \to \infty$

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Spherical Geometry VI

- Separation of variables $u(r,\phi) = G(r)H(\phi)$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial G(r)H(\phi)}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial G(r)H(\phi)}{\partial\phi}\right)=0$$

- Differentiate and divide by $G(r)H(\phi)$

$$\frac{1}{G(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial G(r)}{\partial r}\right)=-\frac{1}{H(\phi)\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial H(\phi)}{\partial\phi}\right)$$

- Set each side equal to a constant k
- Have two (new) ODEs

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Spherical Geometry VII

- Multiply by G(r) and let k = n(n+1) in radial ODE

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial G(r)}{\partial r}\right)=kG(r)=n(n+1)G(r)$$

$$r^2\frac{d^2G(r)}{dr^2}+2r\frac{dG(r)}{dr}-n(n+1)G(r)=0$$

- Linearly independent solutions for G(r): Ar^n and B/r^{n+1} (Proof in charts at end.)
- B = 0 at r = 0 and A = 0 as $r \to \infty$

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Spherical Geometry VIII

- Ordinary differential equation for $H(\phi)$

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial H(\phi)}{\partial\phi}\right)=-kH(\phi)=-n(n+1)H(\phi)$$

- Define $w = \cos(\phi)$ so $\sin^2(\phi) = 1 - w^2$ and

$$\frac{d}{d\phi}=\frac{dw}{d\phi}\frac{d}{dw}=-\sin\phi\frac{d}{dw}$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH(\phi)}{d\phi}\right)=\frac{-\sin\phi}{\sin\phi}\frac{d}{dw}\left[(\sin\phi)\left(-\sin\phi\frac{dH(\phi)}{dw}\right)\right]$$

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Spherical Geometry IX

- Resulting equation for H is known as Legendre's equation)

$$\frac{d}{dw}\left[\left(1-w^2\right)\frac{dH}{dw}\right]=\left(1-w^2\right)\frac{d^2H}{dw^2}-2w\frac{dH}{dw}=-n(n+1)H$$

- Solutions to Legendre's equation known as Legendre polynomials are solutions to a Sturm-Liouville problem over $-1 \le w \le 1$ with a weight function of 1
- If $P_n(w)$ is a Legendre polynomial, and $w = \cos\phi$, we have $H(\phi) = \Sigma_n C_n P_n(\cos\phi)$

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Legendre Polynomials

- The general formula for a Legendre polynomial is shown below

$$P_n(x)=\sum_{m=0}^{\mathrm{int}(n/2)}(-1)^n\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

- Legendre polynomials are orthogonal functions (not normalized)
- See Kreyszig, pp 177-180, 207, and 212

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\delta_{mn}\frac{2}{2m+1}$$

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Some Legendre Polynomials

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = (3x^2 – 1)/2$
- $P_3(x) = (5x^3 – 3x)/2$
- $P_4(x) = (45x^4 – 30x^2 + 3)/8$
- $P_5(x) = (63x^5 – 70x^3 + 15x)/8$
- $P_6(x) = (231x^6 – 315x^4 + 15x^2 – 5)/16$
- $P_{11}(x) = (88179x^{11} – 230945x^9 + 218790 x^7 – 90090x^5 +15015x^3 – 693)/256$

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Legendre Polynomials

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MATLAB Legendre Polynomials

- The MATLAB function legendre(n,x) returns a vector of the Legendre function $P_n^m(x)$ for m = 0, 1, 2, …, n
- This Legendre polynomial is the lowest member of this set, $P_n^0(x)$
- The following MATLAB commands return the Legendre polynomial of order n evaluated at one value of x

```
Pnm = legendre (n, x)
P = Pnm(1,1)
```

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Boundary Condition

- Look at solution that is sum of all eigen-functions to get boundary conditions

$$u(r,\varphi) = \sum_{n=0}^{\infty} C_n P_n(\cos\phi) r^n$$

$$u(R,\varphi) = f(\varphi) = \sum_{n=0}^{\infty} C_n P_n(\cos\phi) R^n$$

- We have to see how we can invoke the orthogonality relationships for the Legendre polynomials with $P_n(\cos \phi)$

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Eigenfunction Expansion

- Eigenfunction expansion is based on the orthogonal relationships

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad \int_{-1}^{1} P_m(x) P_n(x) dx = \delta_{mn} \frac{2}{2m+1}$$

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x) dx}{\int_{-1}^{1} P_n(x) P_n(x) dx} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

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Eigenfunction Expansion II

- Transform our Legendre polynomials $P_n(\cos \phi)$ into $P_n(x)$ where $x = \cos \phi$
- $dx = -\sin \phi \, d\phi$ and x = -1 and 1 correspond to $\phi = \pi$ and 0, respectively
- Make these substitutions in a_n equation

Function has different arguments (ϕ *versus* $\cos \phi$)

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} \tilde{f}(x) P_n(x) dx = -\frac{2n+1}{2} \int_{\pi}^{0} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

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Eigenfunction Expansion III

- Here is boundary condition equation

$$u(R,\varphi) = f(\varphi) = \sum_{n=0}^{\infty} C_n P_n(\cos\phi) R^n$$

- To get orthogonal eigenfunction expansion from last chart we multiply this equation by $P_m(\cos\phi)\sin\phi\, d\phi$ and integrate from 0 to π (reverse integration order eliminates – sign)

$$\int_0^{\pi} f(\varphi) P_m(\cos\phi)\sin\phi\, d\varphi =$$

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Eigenfunction Expansion IV

$$\int_0^{\pi} f(\varphi) P_m(\cos\phi)\sin\phi\, d\varphi =$$

$$\int_0^{\pi}\sum_{n=0}^{\infty} C_n P_m(\cos\phi) P_n(\cos\phi) R^n \sin\phi\, d\varphi$$

$$= R^m C_m \int_0^{\pi} [P_m(\cos\phi)]^2 \sin\phi\, d\varphi = \frac{2R^m C_m}{2m+1}$$

$$C_m = \frac{2m+1}{2R^m}\int_0^{\pi} f(\varphi) P_m(\cos\phi)\sin\phi\, d\varphi$$

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Example, u(R,ϕ) = U_R, a Constant

- Substitute general equation for Legendre polynomials into C_n equation

$$P_n(x) = \sum_{m=0}^{\text{int}(n/2)} (-1)^n \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

$$C_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\varphi) P_n(\cos\phi)\sin\phi\, d\varphi$$

$$= \frac{2n+1}{2R^n}\int_0^{\pi} U_R P_n(\cos\phi)\sin\phi\, d\varphi$$

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Example, u(R,ϕ) = U_R II

- Reverse original transformation
  - Let x = cos ϕ so that dx = -sin ϕ dϕ and x = 1, –1 at f = 0, π

$$C_n = \frac{2n+1}{2R^n}\int_0^{\pi} U_R P_n(\cos\phi)\sin\phi\, d\varphi$$

$$= \frac{2n+1}{2R^n} U_R \int_1^{-1} P_n(x)(-dx) = \frac{2n+1}{2R^n} U_R \int_{-1}^{1} P_n(x) dx$$

$$P_n(x) = \sum_{m=0}^{\text{int}(n/2)} (-1)^n \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

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Example, u(R,ϕ) = U_R III

- Define A_{nm} for Legendre polynomial

$$P_n(x) = \sum_{m=0}^{\text{int}(n/2)} (-1)^n \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} = \sum_{m=0}^{\text{int}(n/2)} A_{nm} x^{n-2m}$$

$$C_n = \frac{2n+1}{2R^n} U_R \int_{-1}^{1} P_n(x) dx$$

$$= \frac{2n+1}{2R^n} U_R \int_{-1}^{1} \sum_{m=0}^{\text{int}(n/2)} A_{nm} x^{n-2m} dx$$

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Example, u(R,ϕ) = U_R IV

$$C_n = \frac{2n+1}{2R^n} U_R \int_{-1}^{1} \sum_{m=0}^{\text{int}(n/2)} A_{nm} x^{n-2m} dx$$

$$= \frac{2n+1}{2R^n} U_R \sum_{m=0}^{\text{int}(n/2)} \left[\frac{A_{nm} x^{n-2m+1}}{n-2m+1}\right]_{-1}^{1}$$

$$= \frac{2n+1}{2R^n} U_R \sum_{m=0}^{\text{int}(n/2)} \frac{A_{nm}\left[1^{n-2m+1} - (-1)^{n-2m+1}\right]}{n-2m+1}$$

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Example, u(R,ϕ) = U_R V

$$C_n = \frac{2n+1}{2R^n} U_R \sum_{m=0}^{\text{int}(n/2)} \frac{A_{nm}\left[1^{n-2m+1} - (-1)^{n-2m+1}\right]}{n-2m+1}$$

- This is zero for odd values of n
- Can show $C_2 = 0$ for $P_2(x) = (3x^2 - 1)/2$

$$C_2 = \frac{2n+1}{2R^n} U_R \int_{-1}^{1} P_2(x)dx =$$

$$\frac{2n+1}{2R^n} U_R \int_{-1}^{1} \frac{3x^2-1}{2} dx = \frac{2n+1}{4R^n} U_R \left[\frac{3x^3}{3} - x\right]_{-1}^{1}$$

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Example, u(R,ϕ) = U_R VI

- Previous result that $C_n = 0$ for n = 2 is also true for all even values of n in C_n based on $P_n(x)$ except for n = 0
- This occurs because $P_0 = 1$ is one of the orthogonal eigenfunctions for Legendre polynomials
- A constant initial condition is orthogonal to all the eigenfunctions except $P_0 = 1$
- So, what is the solution to this problem?

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Example, u(R,ϕ) = U_R VII

- Here is the general solution

$$u(r,\varphi) = \sum_{n=0}^{\infty} C_n P_n(\cos\phi) r^n$$

- But C_n is zero unless n = 0 in which case you can show that $C_n = U_R$
- This gives the solution $u(r,\phi) = U_R$
- This is expected since we said that the solution to Laplace's equation with a constant boundary is a constant

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Vector Calculus

- Important results for Laplace equation
- See notes on vector calculus or chapters nine and ten in Kreyszig for background details not given here
- Results are independent of coordinate system, but Cartesian used for examples
- Introduce gradient and divergence which are vector/scalar functions

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Gradients

- Gradient is a vector in written here in Cartesian space where we have f(x,y,z)
- Definition of gradient $grad\ f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$
- Del operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
- grad f is magnitude and direction of maximum gradient, df/ds
- Grad f is perpendicular to line of constant f

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Physical Gradients

- Gradients of Laplace equation solutions often proportional to flux terms
  - Heat flux and temperature gradient
  - Diffusion flux and mass fraction gradient
  - Velocity and velocity potential in ideal flow
  - Current and electrostatic potential
- If we have a plot of constant potential the lines perpendicular to the potential are flux lines

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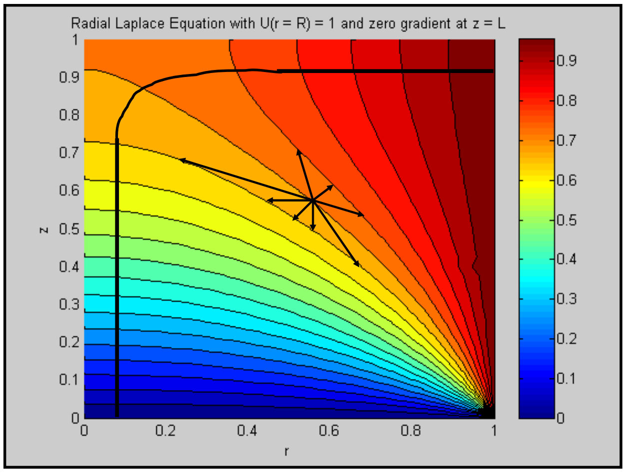

Divergence

- Divergence converts vector, $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, into a scalar written as div **v**
- Definition of divergence $div\ \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
- Del operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
- Gauss divergence theorem (**n** is vector normal to surface, pointing outward)

$$\iiint_{\substack{Enclosed\\Volume}} div\ \mathbf{v}dV = \iint_{Surface} \mathbf{v}\cdot\mathbf{n}dA$$

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$$\iiint_{\substack{Enclosed\\Volume}} div\ \mathbf{q}dV = \iint_{Surface} \mathbf{q}\cdot\mathbf{n}dA$$

- Example of heat flux vector, **q**, (W/m^2)
- **q·n** is component of **q**, normal to surface, dA, flowing outward
- Integrand in surface integral, **q·n**dA is heat flow (watts) flowing out through infinitesimal area, dA
- Surface integral gives total heat flow through surface in outward direction

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Relation to Laplace Equation

- The vector, **v**, may be gradient of a scalar, representing a flux: **v** = -k grad u

$$\iiint_{\substack{Enclosed\\Volume}} div\ \mathbf{v}dV = \iiint_{\substack{Enclosed\\Volume}} div(-k\ grad\ u)dV = \iint_{Surface} \mathbf{v}\cdot\mathbf{n}dA$$

- For constant k

$$\iiint_{\substack{Enclosed\\Volume}} div(grad\ u)dV = \iiint_{\substack{Enclosed\\Volume}} \nabla^2 u dV = -\frac{1}{k}\iint_{Surface} \mathbf{v}\cdot\mathbf{n}dA$$

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Interpretation of $\nabla^2 u = 0$

- When **v** = -k grad u is a flux that is the gradient of a scalar, Laplace's equation for u says that the net inflow of **v** is zero

$$\iiint_{\substack{Enclosed\\Volume}} \nabla^2 u dV = -\frac{1}{k}\iint_{Surface} \mathbf{v}\cdot\mathbf{n}dA = 0$$

- Example of this result shown last week
- Result applies to any problem in any geometry with Laplace's equation

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Complex Variable Basics

- Complex analysis gives insights to Laplace Equation in two dimensions
- Functions of complex variable, z = x + iy: f(z) = u(x,y) + iv(x,y), for example
  - f(z) = z^2 = (x + iy)^2 = x^2 + 2ixy – y^2
  - f(z) = u = x^2 + y^2 and v = 2xy
- What is df/dz? Is it unique?
  - Cauchy-Riemann conditions

$$If\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ then\ \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

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Cauchy-Riemann Example

- f(z) = $z^2 = x^2 - y^2 + 2ixy$ so $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x \quad and \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} = 2y$$

- Satisfies Cauchy-Riemann conditions

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2z$$

$$\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 2x + i2y = 2z$$

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Connection to Laplace Equation

- Take ∂/∂x of first Cauchy-Riemann condition and ∂/∂y of second one and add the results to get Laplace equation

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right] \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right] \Rightarrow \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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Connection to Laplace Equation

- Take ∂/∂y of first Cauchy-Riemann condition and ∂/∂y of second one and subtract for another Laplace equation

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right] \Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right] \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

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Important Result

- If there is a function u(x,y) that satisfies a Laplace equation in two dimensions, there is an associated function v(x,y) that also satisfies Laplace's equation
- The lines of u(x,y) and v(x,y) are mutually perpendicular
- Typically if u is a potential (e.g, temperature, v is a corresponding flux)

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Orthogonal Solutions

- Show that the two solutions u and v are orthogonal at all points
- Consider the gradient of each function

$$\nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} \quad and \quad \nabla v = \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}$$

- Take the dot product of the gradients

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\mathbf{i}\cdot\mathbf{i} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\mathbf{j}\cdot\mathbf{j} + \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right]\mathbf{i}\cdot\mathbf{j} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

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Additional Results

- Treat Laplace equation solutions as complex variable F(z) = u(x,y) + i v(x,y)
- Cauchy theorem for complex integration shows Laplace equation solutions
  - Solutions called harmonic functions
  - Have maximum and minimum on boundary
  - If boundary is a constant at all points then solution is the same constant in region
  - Dirichlet problem has unique solution
- Kreyszig section 18.6 has proofs

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Neumann Problem is not Unique

- Consider the following problem

$$\nabla^2 u = 0 \text{ in a region with } \frac{\partial u}{\partial n} \text{ specified on its boundaries}$$

- If u_1 satisfies the differential equation and boundary conditions,
  – Any other solution = u_1 plus a constant will also satisfy the problem conditions
  – Conclusion: at least part of the boundary must have Dirichlet or third kind of boundary condition

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Conclusions

- Approach to solving Laplace equation is similar to that of diffusion equation
  – Main difference is that second dimension (y) in Laplace equation gives closed boundary instead of open boundary in time
  – Use separation of variables
  – Have eigenfunction solution (sine/cosine, Bessel or other) in one dimension
  – Use eigenfunction expansion to fit condition at one boundary

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Conclusions II

- Use superposition to solve Laplace equation with more than one nonzero boundary
- Additional cylindrical geometry considerations
  – Complex Bessel functions when radial boundary is not eigenfunction solution
  – Must include both Y_0 and J_0 when radial coordinate does not start at zero (must have zero boundary at inner radius)

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Conclusions III

- Results about Laplace's equation from vector analysis and complex variables
- When the gradient of the dependent variable, such as T, in Laplace's equation represents a flux, Laplace's equation says the net outflow is zero
- The maximum and minimum values of a solution to Laplace's equation occur on the boundary so a constant boundary means a constant solution

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Additional Material

- Charts 59 and 60 show proof that solution to radial component of Laplace equation with no θ dependence is $G(r) = Ar^n + Br^{-n-1}$
- If you have any questions about this, please ask. I do not plan to cover this in lecture.

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Spherical Geometry VIIa

- Verify radial solutions for that $G(r) = Ar^n$
- $dG/dr = nAr^{n-1}$; $d^2G/dr^2 = n(n-1)Ar^{n-2}$
- Substitute into radial equation and factor common power of r^n

$$r^2 \frac{d^2G(r)}{dr^2} + 2r\frac{dG(r)}{dr} - n(n+1)G(r) =$$
$$r^2\left(n^2 - n\right)r^{n-2} + 2rnr^{n-1} - n(n+1)r^n =$$
$$r^n\left(n^2 - n + 2n - n^2 - n\right) = 0$$

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Spherical Geometry VIIb

- Verify radial solutions for that $G(r) = Br^{(-n-1)}$
- $dG/dr = -(n+1)Br^{(-n-2)}$; $d^2G/dr^2 = (n+2)(n+1)Br^{(-n-3)} = (n^2 + 3n + 2)\,Br^{(-n-3)}$
- Substitute into radial equation and factor common power of $r^{(-n-1)}$

$$r^2 \frac{d^2G(r)}{dr^2} + 2r\frac{dG(r)}{dr} - n(n+1)G(r) =$$

$$r^2\left(n^2 + 3n + 2\right)r^{-n-3} - 2r\left(n+1\right)r^{-n-2} - n(n+1)r^{-n-1} =$$

$$r^{-n-1}\left(n^2 + 3n + 2 - 2n - 2 - n^2 - n\right) = 0$$

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