Laplace Equation Solutions in Cylindrical Geometry

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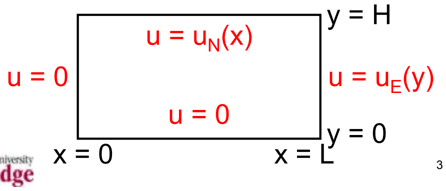
Overview

- Review last class
  - Superposition solutions
  - Introduction to radial coordinates
- Additional solutions of Laplace's equation in radial coordinates
  - Homogenous boundaries in z direction
  - Gradient boundary conditions in the cylinder
  - Hollow cylinder
  - Combinations of boundary conditions
  - Superposition

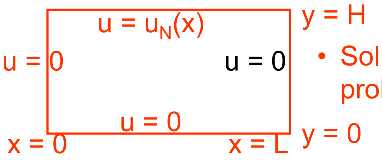
Review Superposition

- Laplace's equation for $0 \le x \le L$ and $0 \le y \le H$ with boundary conditions shown
- Do not have homogenous boundary conditions in any coordinate direction
- Superposition sum of simpler solutions

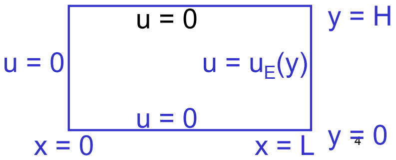


Review Superposition Solution

- Sum two solutions with one nonzero boundary



- Solution to this problem is $u_1(x,y)$
- Solution here is $u_2(x,y)$

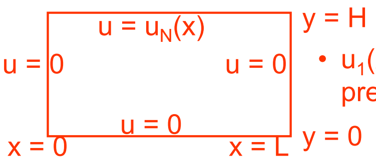


Review General Superposition

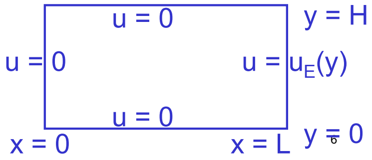
- Can obtain solution for several nonzero boundaries by creating a separate solution for each nonzero boundary
  - Three-dimensional problems can have up to have six separate solutions for u(x,y,z)
- Swap x and y and swap H and L in solution for u(0,y) = u(L,y) = 0 to get solution for y(x,0) = u(x,H) = 0
- Set y = H – y in solution for u(x,H) = $u_b(x)$ to get solution for u(x,0) = $u_b(x)$

Review x and y Swap

- Sum two solutions shown below



- $u_1(x,y)$ found previously
- Swap x and y to get $u_2(x,y)$ from u_1



Review x-y Swap Solution

$$u_1(x,y)=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\sinh(\lambda_n y)\qquad \lambda_n=n\pi/L$$

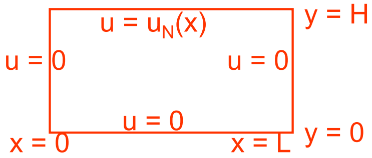
$$u_2(x,y)=\sum_{n=1}^{\infty}B_n\sin(\kappa_n y)\sinh(\kappa_n x)\qquad \kappa_n=n\pi/H$$

$$C_n=\frac{2}{\sinh(\lambda_n H)L}\bullet\int_0^L u_N(x)\sin(\lambda_n x)dx\qquad B_n=\frac{2}{\sinh(\kappa_n L)H}\bullet\int_0^H u_E(y)\sin(\kappa_n y)dy$$

$$u(x,y)=\sum_{n=1}^{\infty}\left[C_n\sin(\lambda_n x)\sinh(\lambda_n y)+B_n\sin(\kappa_n y)\sinh(\kappa_n x)\right]$$

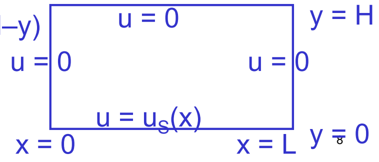
Review Coordinate Transform

- Swap location of one zero boundary



- $u_1(x,y)$ found previously with u_N (x)

- $u_2(x,y) = u_1(x,H–y)$
- Use $u_S(x)$ in place of $u_N(x)$



Review y ← H – y Solution

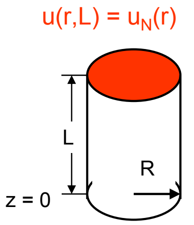
$$u_1(x,y)=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\sinh(\lambda_n y)\qquad \lambda_n=n\pi/L$$

$$C_n=\frac{2}{\sinh(\lambda_n H)L}\int_0^L u_N(x)\sin(\lambda_n x)dx$$

$$u_2(x,y)=\sum_{n=1}^{\infty}B_n\sin(\lambda_n x)\sinh(\lambda_n(H-y))\qquad \lambda_n=n\pi/L$$

$$B_n=\frac{2}{\sinh(\lambda_n H)L}\int_0^L u_S(x)\sin(\lambda_n x)dx$$

Cylindrical Laplace



$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

- Solve Laplace’s equation in a cylinder for u(r,z)
- Zero boundaries at the sides and bottom
  - u(R,z) = 0
  - u(r,0) = 0
- Specified top boundary
  - u(r,L) = $u_N(r)$
- Finite solution at r = 0

What do We Expect?

- Note similarity to radial diffusion equation
  - u is finite at r = 0 for both problems

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0\qquad \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}-\frac{1}{\alpha}\frac{\partial u}{\partial t}=0$$

$$u(R,z)=u_R(z)\qquad u(R,z)=u_R(z)$$

- Separation of variables result for P(r) in Laplace’s equation should be similar to result for diffusion equation
  - Bessel function eigenfunctions in r direction

What do We Expect? II

- Also similar to Laplace equation for u(x,y)

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0\quad u(r,0)=0\quad u(r,L)=u_N(r)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\quad u(x,0)=0\quad u(x,H)=u_N(x)$$

- Separation of variables result for Z(z) in radial equation should be similar to result for Y(y) in rectangular coordinates
  - Hyperbolic sine/cosine solution for Z(z)

Separation of Variables

- Proposed solution u(r,z) = P(r)Z(z)
- Separation of variables and ODE solutions give starting point for solution

$$\frac{1}{rP(r)}\frac{d}{dr}r\frac{dP(r)}{dr} = -\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\lambda^2$$

$$\frac{d^2Z(z)}{dz^2} - \lambda^2 Z(z) = 0 \qquad Z(z) = A\sinh(\lambda z) + B\cosh(\lambda z)$$

$$\frac{d}{dr}r\frac{dP(r)}{dr} + \lambda^2 rP(r) = 0 \qquad P(r) = CJ_0(\lambda r) + DY_0(\lambda r)$$

u(r,z) = P(r)Z(z)

Boundary Conditions

- u(r,0) = 0 for all r requires Z(0) = 0
- Z(0) = 0 = A sinh(0) + B cosh(0) = B = 0
- Finite solution at r = 0 requires D = 0
- u(R,z) = 0 for all z requires P(R) = 0
- P(R) = 0 if $J_0(\lambda R) = 0$ so $\lambda_m R = \alpha_{m0}$, the zeros of J_0
- Solution is sum of all eigenfunctions

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m z) J_0(\lambda_m r) \qquad \lambda_m R = \alpha_{m0}$$

Boundary Condition at z = L

- Radial equation for P(r) is a Sturm-Liouville problem so we use eigenfunction expansion for y = L boundary
  - Region is 0 ≤ r ≤ R and p(r) = r is weight function

$$u_N(r) = u(r,L) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m L) J_0(\lambda_m r)$$

$$C_m = \frac{\int_0^R rJ_0(\lambda_m r)u_N(r)dr}{\sinh(\lambda_m L)\int_0^R r\left[J_0(\lambda_m r)\right]^2 dr} = \frac{\int_0^R rJ_0(\lambda_m r)u_N(r)dr}{\sinh(\lambda_m L)\frac{R^2}{2}\left[J_1(\lambda_m R)\right]^2}$$

Example: $u_N(r)$ = U, a Constant

$$C_m = \frac{\int_0^R rJ_0(\lambda_m r)Udr}{\sinh(\lambda_m L)\frac{R^2}{2}\left[J_1(\lambda_m R)\right]^2} = \frac{2U}{R^2}\frac{\left[\frac{rJ_1(\lambda_m r)}{\lambda_m}\right]_{r=0}^{r=R}}{\sinh(\lambda_m L)\left[J_1(\lambda_m R)\right]^2}$$

$$= \frac{2U\frac{RJ_1(\lambda_m R)}{\lambda_m}}{\sinh(\lambda_m L)R^2\left[J_1(\lambda_m R)\right]^2} = \frac{2U}{\sinh(\lambda_m L)\lambda_m RJ_1(\lambda_m R)}$$

$\lambda_m R = \alpha_{m0}$

- Substitute C_m equation into general solution

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m z) J_0(\lambda_m r)$$

Example: $u_N(r)$ = U, a Constant

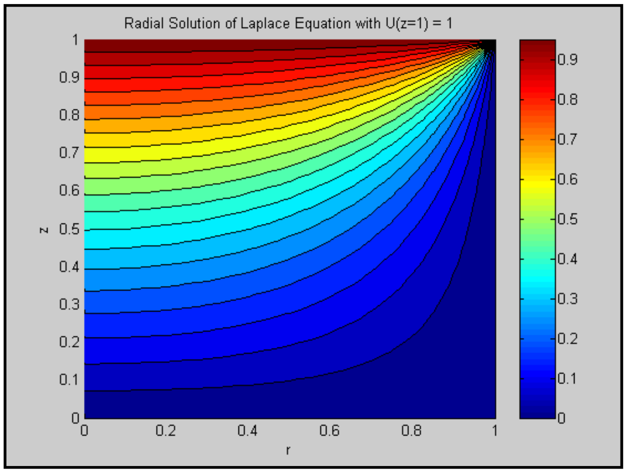
$$C_m = \frac{2U}{\sinh(\lambda_m L)\lambda_m RJ_1(\lambda_m R)} = \frac{2U}{\sinh\left(\frac{\alpha_{m0}}{R}L\right)\frac{\alpha_{m0}}{R}RJ_1\left(\frac{\alpha_{m0}}{R}R\right)}$$

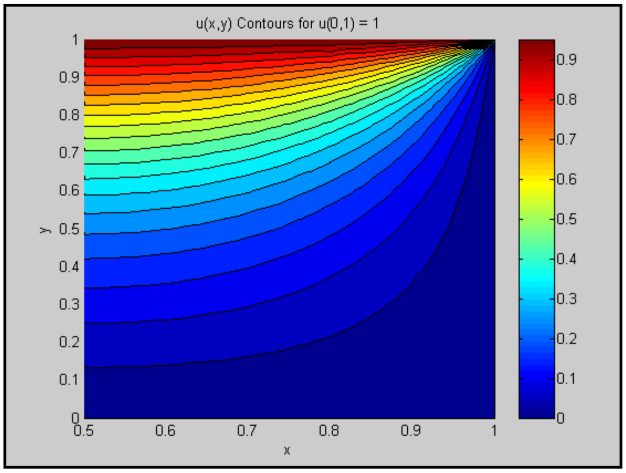
$$C_m = \frac{2U}{\sinh\left(\alpha_{m0}\frac{L}{R}\right)\alpha_{m0}J_1(\alpha_{m0})}$$

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh\left(\alpha_{m0}\frac{z}{R}\right)J_0\left(\alpha_{m0}\frac{r}{R}\right)$$

C_m

$$u(r,z) = \sum_{m=1}^{\infty} \frac{2U}{\alpha_{m0}J_1(\alpha_{m0})}\frac{\sinh\left(\alpha_{m0}\frac{z}{R}\right)}{\sinh\left(\alpha_{m0}\frac{L}{R}\right)}J_0\left(\alpha_{m0}\frac{r}{R}\right)$$





Important Observation

- In solving PDEs by separation of variables, individual terms have the same behavior in any equation (diffusion and Laplace so far)
  - Terms like $\partial^2 u/\partial x^2$, with homogenous boundary conditions give eigenfunctions that are sines and/or cosines
  - Terms like $(1/r)\, \partial\, [r\partial u/\partial r] / \partial r$, with homogenous boundary conditions give Bessel functions as eigenfunctions

Two Boundary Changes

- Laplace’s Equation in two-dimensional cylindrical region $0 \le z \le L$ and $0 \le r \le R$
  - Boundary conditions $u(R,z) = u_R(z)$ Other boundaries: $u(r,0) = 0$, $\partial u/\partial z|_{z=L} = 0$
  - $u(0,z)$ is finite (zero gradient)
- Here we have homogenous boundary conditions in z direction
  - Want Sturm-Liouville solution in this direction to get eigenfunction expansions for nonzero boundary at r = R

Separation of Variables

- Use $u(r,z) = P(r)Z(z)$ to solve equation $$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$
- Pick separation of variables constant to give sine and cosine solution in z

$$-\frac{1}{rP(r)}\frac{d}{dr} r \frac{dP(r)}{dr} = \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = -\lambda^2$$

$$\frac{d^2 Z(z)}{dz^2} + \lambda^2 Z(z) = 0 \qquad Z(z) = A\sin(\lambda z) + B\cos(\lambda z)$$

$$\frac{d}{dr} r \frac{dP(r)}{dr} - \lambda^2 r P(r) = 0 \qquad P(r) = CI_0(\lambda r) + DK_0(\lambda r)$$

Modified Bessel Functions

- $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $K_\nu(x) = i^{-\nu}Y_\nu(ix)$, $i^2 = -1$
- Satisfy modified differential equation

$$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + \nu^2\right)y = 0$$

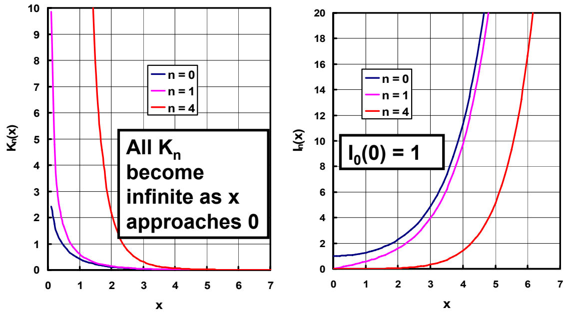
Since $J_\nu \sim x^\nu$, I_ν and K_ν are real

- Equation above transforms to

$$\frac{d}{dz}\left[z\frac{dy}{dz}\right] + \left(\frac{\nu^2}{z} - \lambda^2 z\right)y = 0$$

- Solution is $z = AI_\nu(\lambda x) + BK_\nu(\lambda x)$

Modified Bessel Function Plots



All K_n become infinite as x approaches 0

$I_0(0) = 1$

Boundary Conditions

- $u(r,0) = PZ = 0$ for all r requires $Z(0) = 0$
- $Z(0) = 0 = A \sin(0) + B \cos(0) =$ **B = 0**
- Finite solution at r = 0 requires **D = 0**
- $\partial u/\partial z|_{z=L} = 0$ for all r requires $dZ/dz|_{z=L} = \lambda A \cos(\lambda L) = 0$; **$2\lambda L = n\pi$** (odd integer n)
- P(r) is finite at r = 0 only if B = 0
- Solution is sum of all eigenfunctions

$$u(r,z) = \sum_{m=0}^{\infty} C_m \sin(\lambda_m z) I_0(\lambda_m r) \qquad \lambda_m = (2m+1)\frac{\pi}{2L}$$

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Boundary Condition at r = R

- Equation for Z(z) is a Sturm-Liouville problem; use eigenfunction expansion in $\sin(\lambda_m z)$ for r = R boundary

$$u_R(z) = u(R,z) = \sum_{m=0}^{\infty} C_m \sin(\lambda_m z) I_0(\lambda_m R)$$

$$C_m = \frac{\int_0^L \sin(\lambda_m z) u_R(z)\,dz}{I_0(\lambda_m R)\int_0^L [\sin(\lambda_m z)]^2\,dz} \qquad \lambda_m = (2m+1)\frac{\pi}{2L}$$

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Boundary Condition at r = R II

- Denominator integral in C_m equation

$$\int_0^L [\sin(\lambda_m z)]^2\,dz = \left[\frac{x}{2} - \frac{\sin(\lambda_m z)\cos(\lambda_m z)}{2\lambda_m}\right]_0^L = \frac{L}{2} - \frac{\sin(\lambda_m L)\cos(\lambda_m L)}{2\lambda_m} = \frac{L}{2}$$

- Numerator integral in C_m equation for $u_R = U$

$$\int_0^L \sin(\lambda_m z) U\,dz = -\frac{U\cos(\lambda_m z)}{\lambda_m}\Bigg|_0^L = -U\frac{\cos(\lambda_m L) - 1}{\lambda_m}$$

$$= -U\frac{\cancelto{0}{\cos\left(\frac{(2n+1)\pi}{2}\right)} - 1}{\frac{(2n+1)\pi}{2L}} = \frac{2LU}{(2n+1)\pi}$$

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Boundary Condition at r = R III

- Result for $u_R(z) = U$, a constant $\qquad \lambda_m = (2n+1)\frac{\pi}{2L}$

$$C_m = \frac{\int_0^L \sin(\lambda_m z) U\,dz}{I_0(\lambda_m R)\int_0^L [\sin(\lambda_m z)]^2\,dz} = \frac{\frac{2\cancel{L}U}{(2n+1)\pi}}{I_0(\lambda_m R)\frac{\cancel{L}}{2}} = \frac{4U}{(2n+1)\pi I_0(\lambda_m R)}$$

$$u(r,z) = \frac{4U}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi}{2}\frac{z}{L}\right) I_0\left(\frac{(2n+1)\pi}{2}\frac{r}{L}\right)}{(2n+1) I_0\left(\frac{(2n+1)\pi}{2}\frac{R}{L}\right)}$$

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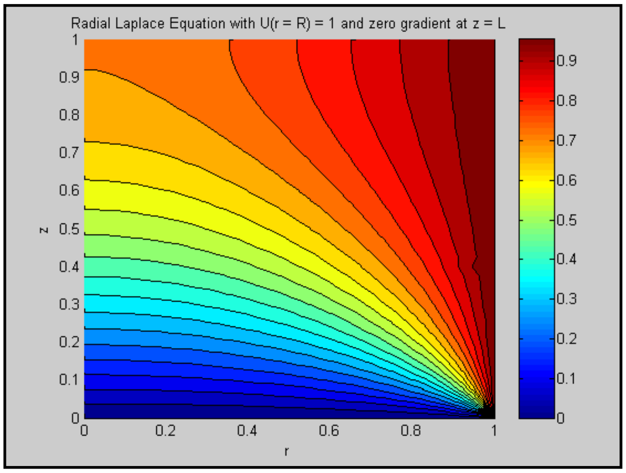

Exercise

- Solve previous problem changing the boundary condition at z = L from a zero gradient to a zero potential

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad u(r,0) = u(r,L) = 0 \quad u(R,z) = u_R(z)$$
$$u \text{ is finite at } r = 0$$

- Pick starting point from previous solution
  $$u(r,z) = [A\sin(\lambda z) + B\cos(\lambda z)][CI_0(\lambda r) + DK_0(\lambda r)]$$
  - D = 0 to eliminate K_0 (= –∞ at r = 0)
  - B = 0 for u(r,0) = 0
  - $\lambda = n\pi/L$ (n an integer) for u(r,L) = 0

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Hollow Cylinder Solution

- Laplace's Equation in two-dimensional region $0 \le z \le L$ and $R_i \le r \le R_0$
  - Boundary conditions $u(R_0,z) = u_R(z)$ with all other boundaries zero: $u(r,0) = u(R_i,z) = u(r,L) = 0$
- $u(r,0) = 0$ and $u(r,L) = 0$ will give Strum-Liouville solution in z direction
  - Expect modified Bessel functions in r direction similar to previous problem
  - Cannot drop K_0 without r = 0 in region

Separation of Variables

- Proposed solution $u(r,z) = P(r)Z(z)$
- Usual separation of variables process and ODE solutions give

$$-\frac{1}{rP(r)}\frac{d}{dr}r\frac{dP(r)}{dr} = \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\lambda^2$$

$$\frac{d^2Z(z)}{dz^2} + \lambda^2 Z(z) = 0 \qquad Z(z) = A\sin(\lambda z) + B\cos(\lambda z)$$

$$\frac{d}{dr}r\frac{dP(r)}{dr} - \lambda^2 rP(r) = 0 \qquad P(r) = CI_0(\lambda r) + DK_0(\lambda r)$$

Boundary Conditions

- $u(r,0) = PZ = 0$ for all r requires $Z(0) = 0$
- $Z(0) = 0 = A \sin(0) + B \cos(0) = B = 0$
- $u(r,L) = PZ = 0$ for all r requires $Z(L) = A\sin(\lambda L) = 0$ so that $\lambda L = n\pi$ (integer n)
- $u(R_i,z) = PZ = 0$ for all z requires $CI_0(\lambda R_i) + DK_0(\lambda R_i) = 0$ so $D = -CI_0(\lambda R_i)/K_0(\lambda R_i)$
- Solution is sum of all eigenfunctions

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sin(\lambda_m z)\left[I_0(\lambda_m r) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m r)\right] \qquad \lambda_m = \frac{m\pi}{L}$$

Boundary Condition at r = R

- Equation for Z(z) is a Sturm-Liouville problem; use eigenfunction expansion in $\sin(\lambda_m z) = \sin(m\pi z/L)$ at $r = R_0$

$$u_R(z) = u(R_0,z) = \sum_{m=1}^{\infty} C_m \sin(\lambda_m z)\left[I_0(\lambda_m R_0) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m R_0)\right]$$

$$C_m = \frac{\int_0^L \sin(\lambda_m z)u_R(z)dz}{\left[I_0(\lambda_m R_0) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m R_0)\right]\int_0^L [\sin(\lambda_m z)]^2 dz}$$

Boundary Condition at r = R II

- Denominator integral in C_m equation

$$\int_0^L [\sin(\lambda_m z)]^2 dz = \left[\frac{x}{2} - \frac{\sin(\lambda_m z)\cos(\lambda_m z)}{2\lambda_m}\right]_0^L = \frac{L}{2} - \frac{\sin(\lambda_m L)\cos(\lambda_m L)}{2\lambda_m} = \frac{L}{2}$$

- Numerator integral in C_m equation for $u_R = U$

$$\int_0^L \sin\left(\frac{m\pi z}{L}\right)Udz = -\left.\frac{UL\cos\left(\frac{m\pi z}{L}\right)}{m\pi}\right|_0^L$$

$$= -UL\frac{\cos(m\pi) - 1}{m\pi} = \begin{cases} \frac{2UL}{m\pi} & odd\ m \\ 0 & even\ m \end{cases}$$

C_m for $u_R(z) = U$

$$C_m = \frac{\int_0^L \sin(\lambda_m z)u_R(z)dz}{\left[I_0(\lambda_m R_0) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m R_0)\right]\int_0^L [\sin(\lambda_m z)]^2 dz}$$

$$= \frac{\frac{2\cancel{L}U}{m\pi}}{\left[I_0(\lambda_m R_0) - \frac{I_0(\lambda_m R_i)}{K_0(\lambda_m R_i)}K_0(\lambda_m R_0)\right]\frac{\cancel{L}}{2}}$$

$\lambda_m = m\pi/L$ for odd m only

Define F_n and G_n to simplify result for u(r,z)

$$F_n = \frac{I_0\left(\frac{(2n+1)\pi R_i}{L}\right)}{K_0\left(\frac{(2n+1)\pi R_i}{L}\right)} \qquad G_n = I_0\left(\frac{(2n+1)\pi R_0}{L}\right) - F_n K_0\left(\frac{(2n+1)\pi R_0}{L}\right)$$

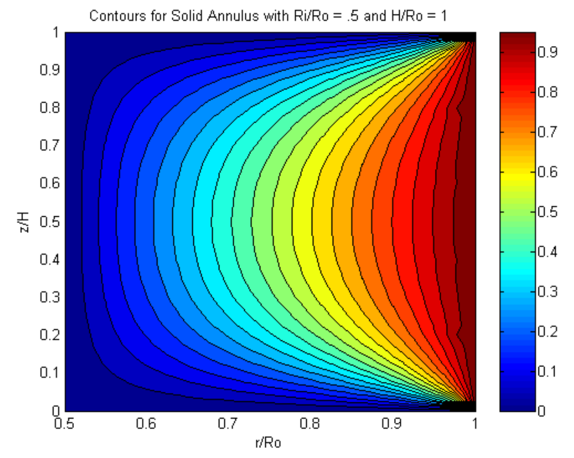
Results for $u_R(z) = U$

$$u(r,z) = \frac{4U}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi z}{L}\right)\left[I_0\left(\frac{(2n+1)\pi r}{L}\right) - F_n K_0\left(\frac{(2n+1)\pi r}{L}\right)\right]}{(2n+1)G_n}$$

$$\frac{u(r,z)}{U} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi z}{2L}\right)\left[I_0\left(\frac{(2n+1)\pi r}{R_0}\frac{R_0}{L}\right) - F_n K_0\left(\frac{(2n+1)\pi r}{R_0}\frac{R_0}{L}\right)\right]}{(2n+1)G_n}$$

$$F_n = \frac{I_0\left(\frac{(2n+1)\pi R_i}{R_0}\frac{R_0}{L}\right)}{K_0\left(\frac{(2n+1)\pi R_i}{R_0}\frac{R_0}{L}\right)} \qquad G_n = I_0\left(\frac{(2n+1)\pi R_0}{L}\right) - F_n K_0\left(\frac{(2n+1)\pi R_0}{L}\right)$$

$$\frac{u}{U} = f\left(\frac{r}{R_0}, \frac{z}{L}, \frac{R_0}{L}, \frac{R_i}{R_0}\right)$$


Contours for Solid Annulus with Ri/Ro = .5 and H/Ro = 1

Another Hollow Cylinder

- Laplace's Equation in two-dimensional region $0 \le z \le L$ and $R_i \le r \le R_0$
  - Boundary conditions $u(r,L) = u_N(z)$ with all other boundaries zero: $u(r,0) = u(R_i,z) = u(R_0,z) = 0$
- Laplace's equation for cylinder with no angular variations

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{\partial^2 u}{\partial z^2} = 0$$

Separation of Variables

- Proposed solution $u(r,z) = P(r)Z(z)$
- Usual separation of variables process and ODE solutions give

$$\frac{1}{rP(r)}\frac{d}{dr}r\frac{dP(r)}{dr} = -\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\lambda^2$$

$$\frac{d^2Z(z)}{dz^2} - \lambda^2 Z(z) = 0 \qquad Z(z) = A\sinh(\lambda z) + B\cosh(\lambda z)$$

$$\frac{d}{dr}r\frac{dP(r)}{dr} + \lambda^2 rP(r) = 0 \qquad P(r) = CJ_0(\lambda r) + DY_0(\lambda r)$$

Boundary Conditions

- $Z(0) = 0 = A \sinh(0) + B \cosh(0) = B = 0$
- $u(R_i,z)$ and $u(R_0,z) = 0$ with $u = PZ$ for all z requires $P(R_i) = P(R_0) = 0$
  - $CJ_0(\lambda R_i) + DY_0(\lambda R_i) = CJ_0(\lambda R_0) + DY_0(\lambda R_0) = 0$

$$\begin{bmatrix} J_0(\lambda R_i) & Y_0(\lambda R_i) \\ J_0(\lambda R_0) & Y_0(\lambda R_0) \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Must have zero determinant zero to avoid trivial solution C = D = 0

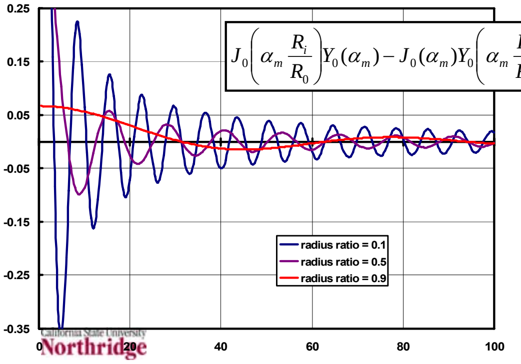
Boundary Conditions

$$Det\begin{bmatrix} J_0(\lambda R_i) & Y_0(\lambda R_i) \\ J_0(\lambda R_0) & Y_0(\lambda R_0) \end{bmatrix} = \begin{matrix} J_0(\lambda R_i)Y_0(\lambda R_0) \\ -J_0(\lambda R_0)Y_0(\lambda R_i) \end{matrix} = 0$$

- This is eigenvalue equation for λ
- Substitute $\alpha_m = \lambda_m R_0$ into this result to show that eigenvalues depend on radius ratio, R_i/R_0

$$J_0\left(\alpha_m\frac{R_i}{R_0}\right)Y_0(\alpha_m) - J_0(\alpha_m)Y_0\left(\alpha_m\frac{R_i}{R_0}\right) = 0$$

Finding Eigenvalues = f(R$_i$/R$_0$)



$$J_0\left(\alpha_m \frac{R_i}{R_0}\right)Y_0(\alpha_m) - J_0(\alpha_m)Y_0\left(\alpha_m \frac{R_i}{R_0}\right) = 0$$

Boundary Conditions

$$\begin{bmatrix} J_0(\lambda R_i) & Y_0(\lambda R_i) \\ J_0(\lambda R_0) & Y_0(\lambda R_0) \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Second row of matrix gives CJ$_0$(λR$_0$) + DY$_0$(λR$_0$) = 0 or C = –DY$_0$(λR$_0$) / J$_0$(λR$_0$)
- Radial solution is P(r) = CJ$_0$(λr) + DY$_0$(λr)
- Combine these to get P(r) = –D [Y$_0$(λR$_0$) / J$_0$(λR$_0$)] J$_0$(λr) + DY$_0$(λr)
- Multiply by J$_0$(λR$_0$) and add m subscript

Boundary Conditions II

$$P(r) = -D\frac{Y_0(\lambda_m R_0)J_0(\lambda_m r)}{J_0(\lambda_m R_0)} + DY_0(\lambda_m r)$$

$$P(r) = -\frac{D}{J_0(\lambda_m R_0)}\left[Y_0(\lambda_m R_0)J_0(\lambda_m r) - J_0(\lambda_m R_0)Y_0(\lambda_m r)\right]$$

- Define new constant: C$_m$ = –D/J$_0$(λ$_m$R$_0$)

$$P_m(r) = C_m P_0(\lambda_m r) = C_m\left[Y_0(\lambda_m R_0)J_0(\lambda_m r) - J_0(\lambda_m R_0)Y_0(\lambda_m r)\right]$$

- Solution is sum of all eignefunctions

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m z)\left[Y_0(\lambda_m R_0)J_0(\lambda_m r) - J_0(\lambda_m R_0)Y_0(\lambda_m r)\right]$$

Boundary Condition at z = L

- Equation for P(r) is a Sturm-Liouville problem; use eigenfunction expansion in P$_m$(r) = Y$_0$(Λ$_m$R$_0$)J$_0$(Λ$_m$r) – J$_0$(Λ$_m$R$_0$) · Y$_0$(Λ$_m$r) at z = L

$$u_N(r) = u(r,L) = \sum_{m=1}^{\infty} C_m \sinh(\lambda_m L)P_0(\lambda_m r)$$

$$C_m = \frac{\int_{R_i}^{R_0} r u_N(r)P_0(\lambda_m r)dr}{\sinh(\lambda_m L)\int_{R_i}^{R_0} r\left[P_0(\lambda_m r)\right]^2 dr} = \frac{\left[\pi\lambda_m J_0(\lambda_m R_i)\right]^2\int_{R_i}^{R_0} r u_N(r)P_0(\lambda_m r)dr}{\sinh(\lambda_m L)2[J_0^2(\lambda_m R_i) - J_0^2(\lambda_m R_0)]}$$

Solution for u$_N$ = U, a Constant

$$C_m = \frac{\left[\pi\lambda_m J_0(\lambda_m R_i)\right]^2\int_{R_i}^{R_0} rUP_0(\lambda_m r)dr}{\sinh(\lambda_m L)2[J_0^2(\lambda_m R_i) - J_0^2(\lambda_m R_0)]}$$

$$= \frac{U\left[\pi\lambda_m J_0(\lambda_m R_i)\right]^2\left[\frac{2[J_0(\lambda_m R_i) - J_0(\lambda_m R_0)]}{\pi\lambda_m^2 J_0(\lambda_m R_i)}\right]}{\sinh(\lambda_m L)2[J_0^2(\lambda_m R_i) - J_0^2(\lambda_m R_0)]}$$

$$= \frac{\pi U J_0(\lambda_m R_i)}{\sinh(\lambda_m L)[J_0(\lambda_m R_i) + J_0(\lambda_m R_0)]}$$

Integrals from Carslaw and Jaeger, *Conduction of Heat in Solids*, Oxford, 1959.

$$\boxed{u(r,z) = U\pi\sum_{m=1}^{\infty}\frac{\sinh(\lambda_m z)}{\sinh(\lambda_m L)}\frac{J_0(\lambda_m R_i)P_0(\lambda_m r)}{J_0(\lambda_m R_i) + J_0(\lambda_m R_0)}}$$

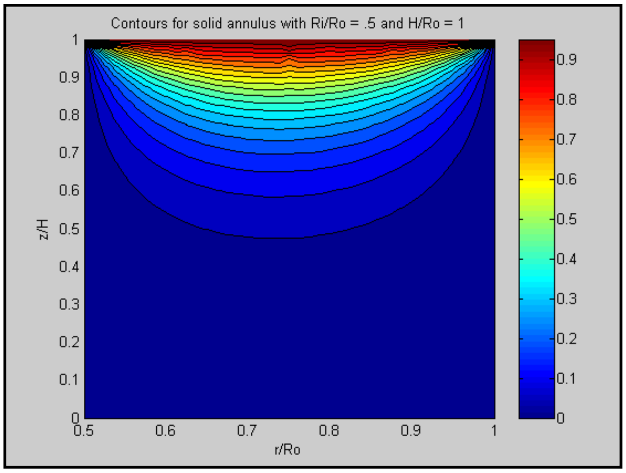
Solution for u$_N$ = U, a Constant

$$u(r,z) = U\pi\sum_{m=1}^{\infty}\frac{\sinh(\lambda_m z)}{\sinh(\lambda_m L)}\frac{J_0(\lambda_m R_i)P_0(\lambda_m r)}{J_0(\lambda_m R_i) + J_0(\lambda_m R_0)}$$

- Convert to dimensionless form; we previously showed that α$_m$ = f(R$_i$/R$_0$)

$$\frac{u(r,z)}{U} = \pi\sum_{m=1}^{\infty}\frac{\sinh\left(\alpha_m \frac{L}{R_0}\frac{z}{L}\right)}{\sinh\left(\alpha_m \frac{L}{R_0}\right)}\frac{J_0\left(\alpha_m \frac{R_i}{R_0}\right)P_0\left(\alpha_m \frac{r}{R_0}\right)}{J_0\left(\alpha_m \frac{R_i}{R_0}\right) + J_0(\alpha_m)}$$

U(r,z)/U depends on z/L, r/R$_0$, R$_i$/R$_0$, and L/R



Possible Cylinder Problems

- Can have solid or hollow cylinder
- Can use superposition to handle more than one nonhomogenous boundary
- Can have boundary conditions in terms of potential, gradients, or a linear combination of the two
- Functions used for solutions depend on the direction for the homogenous boundary

Possible Cylinder Problems II

- Homogenous radial boundaries give radial eigenfunction solution as Bessel functions and vertical solution as hyperbolic sine and cosine
- Homogenous vertical boundaries give eigenfunction solution as sines and cosines and radial solution as modified Bessel functions
- For solid cylinders with r = 0 in domain, Y_0 or K_0 are not present