Laplace Equation Solutions

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Overview

- Review last class
  - Laplace equation solutions for homogenous boundary conditions on three boundaries
- Solutions of Laplace's equation for more than one nonzero boundaries
  - Superposition solutions
  - Superposition for gradient and other boundary conditions
- Cylindrical coordinates

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Review Laplace's Equation

- Used to express equilibrium fields of engineering variable like temperature, species concentration, electrostatic potential and ideal fluid flow
- Written in general coordinates as $\nabla^2 u = 0$

$$Cartesian \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$Cylindrical \qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$Sphere \ \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$$

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Similar to Diffusion Equation

Diffusion	Laplace
$\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$
$0 \le x \le L \quad t \ge 0$	$0 \le x \le L \quad 0 \le y \le H$
$u(0,t) = 0$	$u(0,y) = 0$
$u(L,t) = 0$	$u(L,y) = 0$
$u(r,t) = u_0(r)$	$u(x,H) = u_0(r)$
open boundary in t	$u(x,0) = 0$

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Separation of Variables

Diffusion	Laplace
$-\lambda^2 = \frac{1}{\alpha T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$	$-\lambda^2 = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$
$T(t) = Ae^{-\lambda^2 \alpha t}$	$Y(y) = A\sinh(\lambda y) + D\cosh(\lambda y)$

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x = 0,L Boundary Conditions

Diffusion	Laplace
$X(x) = B\sin(\lambda x) + C\cos(\lambda x)$	$X(x) = B\sin(\lambda x) + C\cos(\lambda x)$
$C = 0$ for $u(0,t) = 0$	$C = 0$ for $u(0,y) = 0$
$\lambda = \frac{n\pi}{L}$ for $u(L,t) = 0$	$\lambda = \frac{n\pi}{L}$ for $u(L,y) = 0$
$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$	$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$

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y = 0 Boundary Condition

Diffusion	Laplace
No equivalent condition because of open boundary in time	$Y(0) = A\sinh(\lambda 0) + D\cosh(\lambda 0) = 0$ $Must\ have\ D = 0$ $Y_n(y) = A_n \sinh\left(\frac{n\pi y}{L}\right)$

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General Solution; Fitted Condition

Diffusion	Laplace
$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \alpha t} \sin(\lambda_n x)$	$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\lambda_n y)\sin(\lambda_n x)$
$\lambda_n = n\pi / L$	$\lambda_n = n\pi / L$
$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x)$	$u(x,H) = u_N(x) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r)\sinh(\lambda_n H)$

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Eigenvalue Expansion for C_n

Diffusion	Laplace
$\dfrac{\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx} = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$\dfrac{\int_0^L u_N(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{n\pi H}{L}\right)\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx} = \dfrac{\frac{2}{L}\int_0^L u_N(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{n\pi H}{L}\right)}$

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Review Constant Boundary

- If $u_N(x) = U$, the solution for $u(x,y)$ is

C_m

$$u(x,y) = \frac{4U}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- u/U = depends on x/L, y/L and H/L
- Plot on next page for H/L = 1

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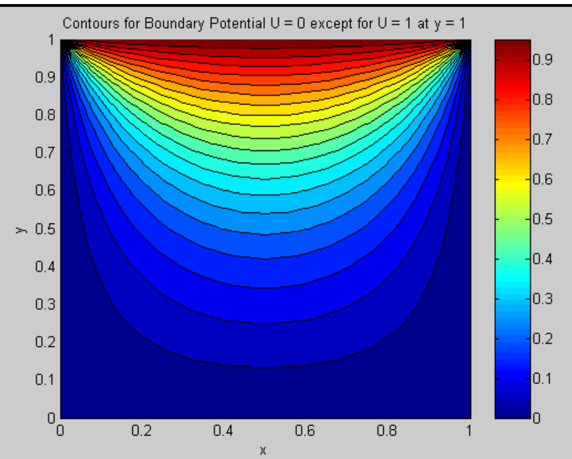

Review Gradients as Fluxes

- Laplace and diffusion equation are based on conservation of fluxes which are (negative) gradients of potential
- Laplace equation gives equilibrium where net flux from region should be zero
- Provided notes showing that net outflow is zero for first Laplace solution

$$Net\ outflow = \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=0} dy - \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=L} dy + \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx - \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx = 0$$

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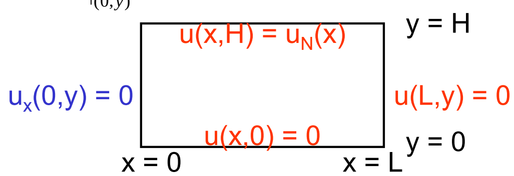
Review Gradient Boundary

- Zero gradient at x = 0

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad 0\le x\le L, \quad 0\le y\le H$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,y)}=u(L,y)=u(x,0)=0 \qquad u(x,H)=u_N(x)$$



Review Zero Gradient Solution

- General solution

$$u(x,y)=\sum_{n=0}^{\infty}C_n\cos(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n=\frac{(2n+1)\pi}{2L}$$

$$C_n=\frac{2}{\sinh(\lambda_n H)L}\int_0^L u_N(x)\cos(\lambda_n x)dx \qquad \lambda_n=\frac{(2n+1)\pi}{2L}$$

- Solution for $u_N(x)$ = U, a constant

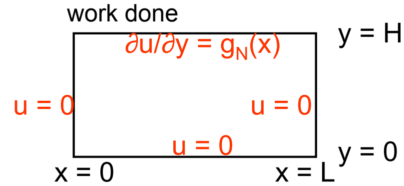
$$u(x,y)=\frac{2U}{L}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(\lambda_n x)\sinh(\lambda_n y)}{\lambda_n\sinh(\lambda_n H)} \qquad \lambda_n=\frac{(2n+1)\pi}{2L}$$

Rectangular Laplace Summary

- **Problem:** homogenous boundary conditions except at y = H
- Separation of variables solution gives u(x,y) = X(x)Y(y) = [Asin(λx) + Bcos(λx)] • [Csinh(λy) + Dcosh(λy)] (Start here!)
- Boundary conditions at x = 0 and x = L give A or B and eigenvalue, λ
- Eigenfunction expansion at y = H gives coefficients in infinite series solution of all eigenfunctions

Exercise

- Solve problem from last class with boundary condition at y = H changed to a gradient
- Use previous solution of similar problem as starting point – in this case have most of the work done



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$u(x,y)=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n={n\pi}/{L}$$

Exercise Solution

- Start with potential solution

$$u(x,y)=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n={n\pi}/{L}$$

- Take the gradient

$$\frac{\partial u(x,y)}{\partial y}=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\lambda_n\cosh(\lambda_n y) \qquad \lambda_n={n\pi}/{L}$$

- Apply y = H boundary condition

$$g_N(x)=\frac{\partial u(x,H)}{\partial y}=\sum_{n=1}^{\infty}C_n\sin(\lambda_n x)\lambda_n\cosh(\lambda_n H) \qquad \lambda_n={n\pi}/{L}$$

Exercise Solution II

- Get eigenfunction expansion

$$C_n=\frac{2}{\cosh(\lambda_n H)\lambda_n L}\int_0^L g_N(x)\sin(\lambda_n x)dx \qquad \lambda_n=\frac{n\pi}{L}$$

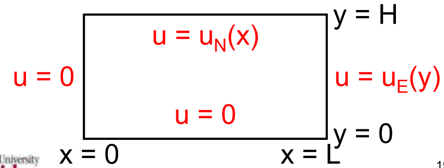
- For constant gradient, $g_N(x) = G_N$

$$C_n=\frac{2\int_0^L G_N\sin(\lambda_n x)dx}{\cosh(\lambda_n H)\lambda_n L}=\frac{\left.-\frac{2G_N}{\lambda_n}\cos(\lambda_n x)\right|_0^L}{\cosh(\lambda_n H)\lambda_n L}=\frac{\frac{2G_N}{\lambda_n}\left[1-(-1)^n\right]}{\cosh(\lambda_n H)\lambda_n L}$$

$$u(x,y)=4G_N\sum_{n=1,3,5,\ldots}\frac{\sin(\lambda_n x)\sinh(\lambda_n y)}{\cosh(\lambda_n H)\lambda_n^2 L} \qquad \lambda_n={n\pi}/{L}$$

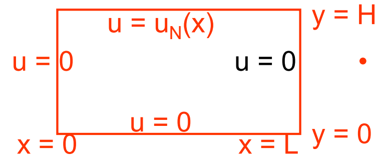
More than One Nonzero Boundary

- Laplace's equation for $0 \le x \le L$ and $0 \le y \le H$ with boundary conditions shown
- Do not have homogenous boundary conditions in any coordinate direction
- Solution is sum of two simpler solutions



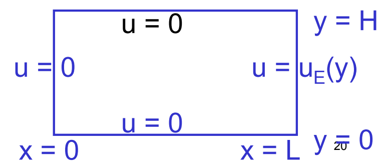
Superposition Solution

- Solution is $u(x,y) = u_1(x,y) + u_2(x,y)$



- $u_1(x,y)$ has $u(L,y) = 0$

- $u_2(x,y)$ has $u(H,x) = 0$

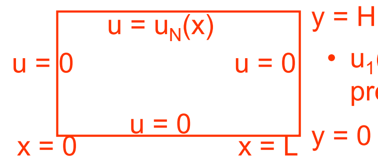


Superposition Solution II

- $u(x,y) = u_1(x,y) + u_2(x,y)$
- u_1 and u_2 satisfy Laplace's equation and their sum satisfies boundary conditions
  - $u(x,0) = u_1(x,0) + u_2(x,0) = 0 + 0 = 0$
  - $u(x,H) = u_1(x,H) + u_2(x,H) = u_N + 0 = u_N$
  - $u(0,y) = u_1(0,y) + u_2(0,y) = 0 + 0 = 0$
  - $u(L,y) = u_1(L,y) + u_2(L,y) = 0 + u_E = u_E$
- Solution for u_2 is same as solution for u_1 with x and y (and L and H) interchanged

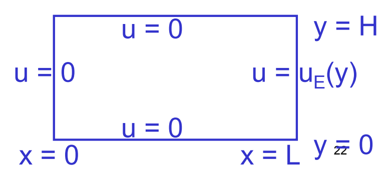
Superposition Solution III

- Sum two solutions shown below



- $u_1(x,y)$ found previously

- Swap x and y to get $u_2(x,y)$ from u_1



Superposition Solution IV

- $u(x,y) = u_1(x,y) + u_2(x,y)$

$$u_1(x,y) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n = {n\pi}/{L}$$

$$C_n = \frac{2}{\sinh(\lambda_n H)L}\int_0^L u_N(x)\sin(\lambda_n x)dx$$

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin(\kappa_n y)\sinh(\kappa_n x) \qquad \kappa_n = {n\pi}/{H}$$

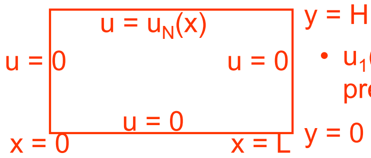
$$B_n = \frac{2}{\sinh(\kappa_n L)H}\int_0^H u_E(y)\sin(\kappa_n y)dy$$

General Superposition

- Can obtain solution for several nonzero boundaries by creating a separate solution for each nonzero boundary
- A three-dimensional problem, with non-zero conditions on each face would have six separate solutions for u(x,y,z)
- A solution for the nonzero boundary at, for example, y = 0 instead of y = H can be handled by a coordinate transform
- Can also handle gradient boundaries

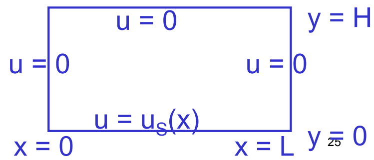
Solution at Opposite Boundary

- Move zero boundary from y = 0 to y = H



- $u_1(x,y)$ found previously

- Define
  z = H – y
  z=0 at y = H
  z=H at y = 0



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Same PDE with y = H – z

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

u(0,y) = u(L,y) = 0
u(x,0) = 0; u(x,H) = $u_B(x)$
New coordinate: y = H – z

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial u}{\partial y}=\frac{\partial}{\partial (H-z)}\frac{\partial u}{\partial (H-z)}=\frac{\partial}{\partial (-z)}\frac{\partial u}{\partial (-z)}=\frac{\partial u^2}{\partial z^2}$$

u(0,z) = u(L,z) = 0
u(x,y = 0) = u(x,z = H) = 0
u(x,y = H) = u(x,z = 0) = $u_B(x)$

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Opposite Boundary Solution

$$u_1(x,y)=\sum_{n=1}^{\infty} C_n \sin(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n = n\pi/L$$

$$C_n=\frac{2}{\sinh(\lambda_n H)L}\int_0^L u_N(x)\sin(\lambda_n x)dx$$

$$u_2(x,y)=\sum_{n=1}^{\infty} B_n \sin(\lambda_n x)\sinh(\lambda_n (H-y)) \qquad \lambda_n = n\pi/L$$

$$B_n=\frac{2}{\sinh(\lambda_n H)L}\int_0^L u_S(x)\sin(\lambda_n x)dx$$

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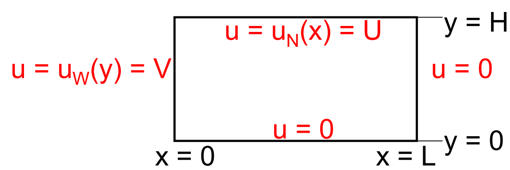
Superposition Example

- Problem with two nonzero boundaries
- Solve by superposition
- Can use original solution with three zero boundaries and u = $u_N(x)$



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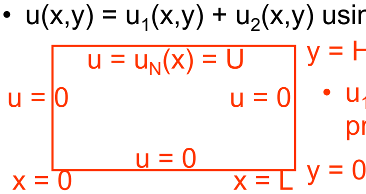
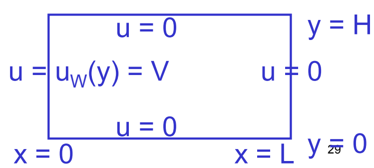
Superposition and Transform

- u(x,y) = $u_1(x,y)$ + $u_2(x,y)$ using swap



- $u_1(x,y)$ found previously

- $u_2(x,y)$ with x' = L - x



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Superposition Solution

- $u_1(x,y)$ from chart 30 of February 17 class

$$u_1(x,y)=\frac{4U}{L}\sum_{n=0}^{\infty}\frac{\sin(\lambda_n x)\sinh(\lambda_n y)}{\lambda_n \sinh(\lambda_n H)} \qquad \lambda_n=\frac{(2n+1)\pi}{L}$$

- Swap x and y and use L – x for x

$$u_2(x,y)=\frac{4V}{H}\sum_{n=0}^{\infty}\frac{\sin(\kappa_n y)\sinh[\kappa_n (L-x)]}{\kappa_n \sinh(\kappa_n L)} \qquad \kappa_n=\frac{(2n+1)\pi}{H}$$

$$u(x,y)=4\sum_{n=0}^{\infty}\left\{\frac{U}{L}\frac{\sin(\lambda_n x)\sinh(\lambda_n y)}{\lambda_n \sinh(\lambda_n H)}+\frac{V}{H}\frac{\sin(\kappa_n y)\sinh[\kappa_n (L-x)]}{\kappa_n \sinh(\kappa_n L)}\right\}$$

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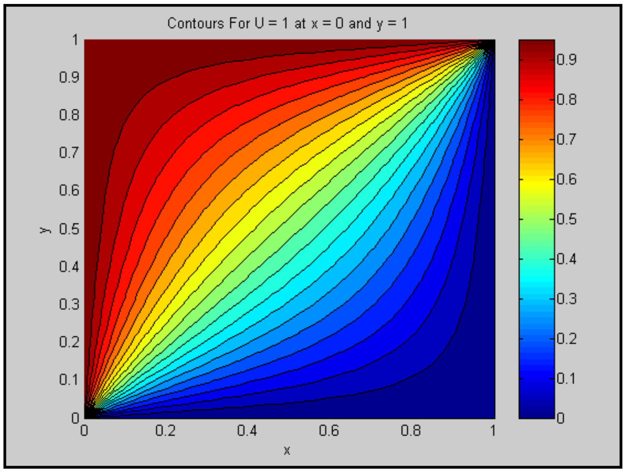


Gradient Superposition

- What if we have specified, nonzero gradients at more than one boundary?
- Example boundary conditions: $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ nonzero as shown
- Other boundaries homogenous

$u_y = G_N(x)$ (at $y = H$)

$au + bu_x = 0$ (at $x = 0$)

$u_x = G_E(y)$ (at $x = L$)

$\alpha u + \beta u_y = 0$ (at $y = 0$)

Gradient Superposition II

- Solution components have ***zero gradients***

$u_1\ (x,y)$

$u_y = G_N(x)$

$au + bu_x = 0$ $\qquad$ ***$u_x = 0$***

$\alpha u + \beta u_y = 0$

$u_2\ (x,y)$

$u_y = 0$

$au + bu_x = 0$ $\qquad$ $u_x = G_E(y)$

$\alpha u + \beta u_y = 0$

Gradient Superposition III

- $u(x,y) = u_1(x,y) + u_2(x,y)$
- u_1 and u_2 satisfy Laplace's equation and their sum satisfies boundary conditions
  - $\alpha u(x,0) + \beta u_y(x,0) = \alpha u_1(x,0) + \beta(u_1)_y(x,0) + \alpha u_2(x,0) + \beta(u_2)_y(x,0) = 0 + 0 = 0$
  - $au(0,y) + bu_x(0,y) = au_1(0,y) + b(u_1)_x(0,y) + au_2(0,y) + b(u_2)_x(0,y) = 0 + 0 = 0$
  - $u_y(x,H) = (u_1)_y(x,H) + (u_2)_y(x,H) = G_N + 0 = G_N$
  - $u_x(L,y) = (u_1)_x(L,y) + (u_2)_x(L,y) = 0 + G_E = G_E$
- Solution for u_2 is same as solution for u_1 with x and y (and L and H) interchanged

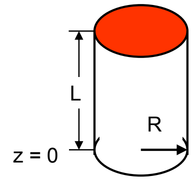
Two Dimensional Cylinder

- Laplace's Equation in two-dimensional region $0 \le z \le L$ and $0 \le r \le R$
  - $u(r,z)$ is temperature, concentration, etc.
  - Top boundary condition $u(r,L) = u_N(r)$
  - Other boundaries: $u(r,0) = u(R,z) = 0$
  - $u(0,z)$ is finite
- Laplace's equation for cylinder with no angular variations

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{\partial^2 u}{\partial z^2} = 0$$

Cylinder Problem Diagram

$u(r,L) = u_N(r)$



- Solve Laplace's equation in a cylinder for $u(r,z)$
- Zero boundaries at the sides and bottom
  - $u(R,z) = 0$
  - $u(r,0) = 0$
- Specified top boundary
  - $u(r,L) = u_N(r)$
- Finite solution at $r = 0$

What do We Expect?

- Note similarity to radial diffusion equation
  - u is finite at r = 0 for both problems

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \qquad \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}-\frac{1}{\alpha}\frac{\partial u}{\partial t}=0$$
$$u(R,z)=u_R(z) \qquad u(R,z)=u_R(z)$$

- Separation of variables result for P(r) in Laplace's equation should be similar to result for diffusion equation
  - Bessel function eigenfunctions in r direction

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What do We Expect? II

- Also similar to Laplace equation for u(x,y)

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \quad u(r,0)=0 \quad u(r,L)=u_N(r)$$
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad u(x,0)=0 \quad u(x,H)=u_N(x)$$

- Separation of variables result for Z(z) in radial equation should be similar to result for Y(y) in rectangular coordinates
  - Hyperbolic sine/cosine solution for Z(z)

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Separation of Variables

- Proposed solution u(r,z) = P(r)Z(z)
- Separation of variables and ODE solutions give starting point for solution

$$\frac{1}{rP(r)}\frac{d}{dr}r\frac{dP(r)}{dr}=-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=-\lambda^2$$

$$\frac{d^2Z(z)}{dz^2}-\lambda^2 Z(z)=0 \qquad Z(z)=A\sinh(\lambda z)+B\cosh(\lambda z)$$

$$\frac{d}{dr}r\frac{dP(r)}{dr}+\lambda^2 rP(r)=0 \qquad P(r)=CJ_0(\lambda r)+DY_0(\lambda r)$$

u(r,z) = P(r)Z(z)

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Boundary Conditions

- u(r,0) = 0 for all r requires Z(0) = 0
- Z(0) = 0 = A sinh(0) + B cosh(0) = B = 0
- Finite solution at r = 0 requires D = 0
- u(R,z) = 0 for all z requires P(R) = 0
- P(R) = 0 if $J_0(\lambda R) = 0$ so $\lambda_m R = \alpha_{m0}$, the zeros of J_0
- Solution is sum of all eigenfunctions

$$u(r,z)=\sum_{m=1}^{\infty}C_m\sinh(\lambda_m z)J_0(\lambda_m r) \qquad \lambda_m R=\alpha_{m0}$$

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Boundary Condition at z = L

- Radial equation for P(r) is a Sturm-Liouville problem so we use eigenfunction expansion for y = L boundary
  - Region is 0 ≤ r ≤ R and p(r) = r is weight function

$$u_N(r)=u(r,L)=\sum_{m=1}^{\infty}C_m\sinh(\lambda_m L)J_0(\lambda_m r)$$

$$C_m=\frac{\int_0^R rJ_0(\lambda_m r)u_N(r)dr}{\sinh(\lambda_m L)\int_0^R r[J_0(\lambda_m r)]^2dr}=\frac{\int_0^R rJ_0(\lambda_m r)u_N(r)dr}{\sinh(\lambda_m L)\frac{R^2}{2}[J_1(\lambda_m R)]^2}$$

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Example: $u_N(r)$ = U, a Constant

$$C_m=\frac{\int_0^R rJ_0(\lambda_m r)Udr}{\sinh(\lambda_m L)\frac{R^2}{2}[J_1(\lambda_m R)]^2}=\frac{2U}{R^2}\frac{\left[\frac{rJ_1(\lambda_m r)}{\lambda_m}\right]_{r=0}^{r=R}}{\sinh(\lambda_m L)[J_1(\lambda_m R)]^2}$$

$$=\frac{2U\frac{RJ_1(\lambda_m R)}{\lambda_m}}{\sinh(\lambda_m L)R^2[J_1(\lambda_m R)]^2}=\frac{2U}{\sinh(\lambda_m L)\lambda_m RJ_1(\lambda_m R)}$$

$\boxed{\lambda_m R = \alpha_{m0}}$

- Substitute C_m equation into general solution

$$u(r,z)=\sum_{m=1}^{\infty}C_m\sinh(\lambda_m z)J_0(\lambda_m r)$$

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Example: $u_N(r)$ = U, a Constant

$$C_m = \frac{2U}{\sinh(\lambda_m L)\lambda_m R J_1(\lambda_m R)} = \frac{2U}{\sinh\left(\frac{\alpha_{m0}}{R}L\right)\frac{\alpha_{m0}}{R}RJ_1\left(\frac{\alpha_{m0}}{R}R\right)}$$

$$C_m = \frac{2U}{\sinh\left(\alpha_{m0}\frac{L}{R}\right)\alpha_{m0}J_1(\alpha_{m0})}$$

$$u(r,z) = \sum_{m=1}^{\infty} C_m \sinh\left(\alpha_{m0}\frac{z}{R}\right)J_0\left(\alpha_{m0}\frac{r}{R}\right)$$

$$u(r,z) = \sum_{m=1}^{\infty} \underbrace{\frac{2U}{\alpha_{m0}J_1(\alpha_{m0})}\frac{1}{\sinh\left(\alpha_{m0}\frac{L}{R}\right)}}_{C_m}\sinh\left(\alpha_{m0}\frac{z}{R}\right) J_0\left(\alpha_{m0}\frac{r}{R}\right)$$

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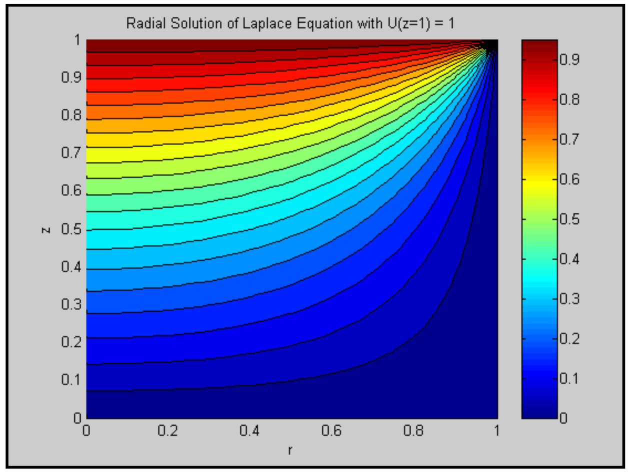

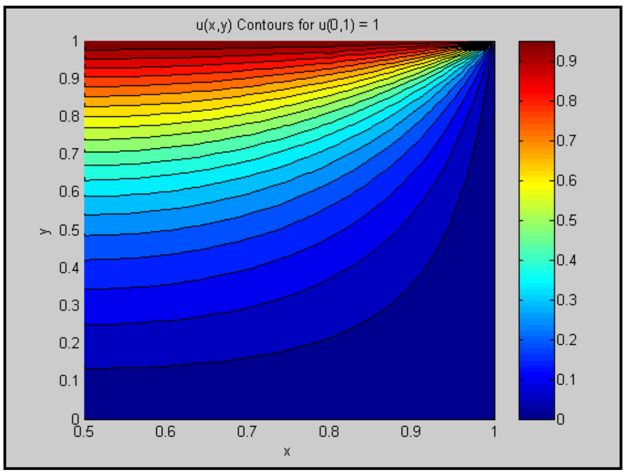

Important Observation

- In solving PDEs by separation of variables, individual terms have the same behavior in any equation (diffusion and Laplace so far)
  - Terms like $\partial^2 u/\partial x^2$, with homogenous boundary conditions give eigenfunctions that are sines and/or cosines
  - Terms like $(1/r)\,\partial\,[r\partial u/\partial r]\,/\,\partial r$, with homogenous boundary conditions give Bessel functions as eigenfunctions

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Superposition Summary

- Need superposition when we have more than one nonzero boundary
- Get components that have all boundaries zero except one
- Choose zero boundaries for potential or gradient as appropriate
- Can usually get solutions of superposition components by simple variable exchanges or transformations

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