Laplace's Equation

Larry Caretto
Mechanical Engineering 501B

Seminar in Engineering Analysis

February 9, 2009

California State University Northridge

Outline

- Review last class
  - Used Bessel functions to solve radial diffusion equation
- Laplace's equation in two-dimensional rectangular coordinates
- Solution by separation of variables
- Result has sine, cosine, and hyperbolic sine and cosine
- Sine and cosine eigenfunction expansion

California State University Northridge 2

Review Radial Diffusion

- Governs diffusion (heat conduction) in cylinder for t ≥ 0 and 0 ≤ r ≤ R
  - u(r,t) is temperature, species concentration
  - Initial condition u(r,0) = u_0(r)
  - u(0,t) is finite and u(R,t) = u_R
  - v(r,t) = u(r,t) – u_R has v(R,0) = 0
- Differential equation for both u and v
  - v is Sturm-Liouville

$$\frac{\partial u}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r}$$

California State University Northridge 3

Review Separation of Variables

- Assume v(r,t) = P(r)T(t)

$$\frac{\partial u}{\partial t} = \frac{\partial[P(r)T(t)]}{\partial t} = P(r)\frac{\partial T(t)}{\partial t} =$$

$$\alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial [P(r)T(t)]}{\partial r} = \alpha T(t) \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial P(r)}{\partial r}$$

- Divide by αP(r)T(t)

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{P(r)}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial P(r)}{\partial r} = -\lambda^2$$

California State University Northridge 4

Review General Solution v(r,t)

- Have exponential solution in time and Bessel's equation for radial function

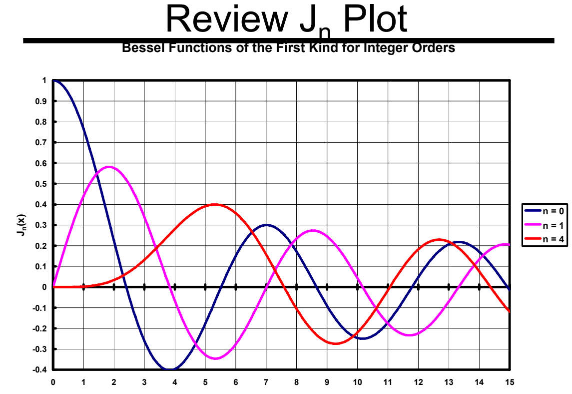
$$\frac{dT(t)}{dt} = -\lambda^2 \alpha T(t) \qquad T(t) = Ae^{-\lambda^2 \alpha t}$$

$$\frac{d}{dr} r \frac{dP(r)}{dr} + \lambda^2 r P(r) = 0$$

- This is form of Bessel's equation
- v(r,t) = T(t)P(r) where equation is P(r) = BJ_0(λr) + CY_0(λr)
  - J_0 and Y_0 are (order zero) Bessel functions of first and second kind, respectively

California State University Northridge 5

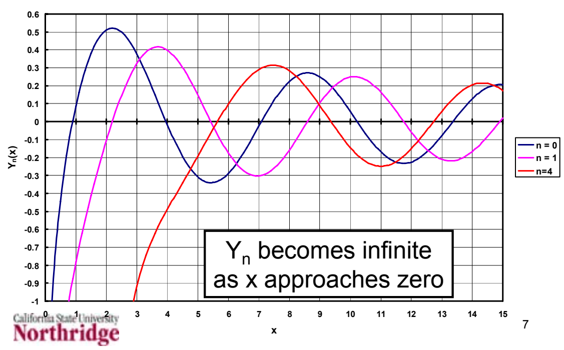
Review J_n Plot



California State University Northridge 6

Review Y_n Plot

Bessel Functions of the Second Kind of Integer Order



California State University Northridge 7

Review Radial Solution

- Radial diffusion equation for P(r) $\frac{d}{dr} r \frac{dP(r)}{dr} + \left(\frac{0}{r} + \lambda^2 r \right) P(r) = 0$ p(r) = r
- Transformed Bessel equation $\frac{d}{dz}\left[z \frac{dy}{dz} \right] + \left(-\frac{\nu^2}{z} + k^2 z \right) y = 0$
- Solution to second (general ν) equation is y = $AJ_\nu(kz) + BY_\nu(kz)$
- Radial diffusion with ν = 0 and p(r) = r is P(r) = $AJ_0(\lambda r) + BY_0(\lambda r)$

California State University Northridge 8

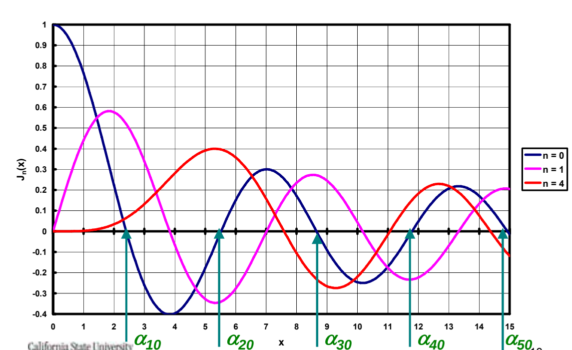
Review Radial Solution II

- v(r,t) = T(t)P(r) where T(t) is exponential decay P(r) = $AJ_0(\lambda r) + BY_0(\lambda r)$
- Solution applies in region to $0 \le r \le R$
- Must have P(r) finite at r = 0
- Since $Y_0(r) \to -\infty$ as $r \to 0$ we must have B = 0 for P(r) to be finite at r = 0
- Condition P(R) = 0 requires $J_0(\lambda R) = 0$
- Eigenvalue solutions are $J_0(\lambda_n r)$ where $\lambda_n R = \alpha_{n0}$ which are zeros of J_0

California State University Northridge 9

Zeros of J_0

Bessel Functions of the First Kind for Integer Orders



California State University Northridge 10

Review Radial Solution III

- Solution is sum of all eigenfunctions

$$u(r,t) = \sum_{m=1}^{\infty} C_m e^{-\lambda_m^2 \alpha t} J_0(\lambda_m r) + u_R \qquad \lambda_m R = \alpha_{m0}$$

- Initial condition, v(r,0) = $u_0(r) - u_R$ gives C_m by eigenfunction expansion

$$C_m = \frac{\int_0^R r J_0(\lambda_m r)\left(u_0(r) - u_R\right) dr}{\int_0^R r \left[J_0(\lambda_m r)\right]^2 dr} = \frac{\int_0^R r J_0(\lambda_m r)\left(u_0(r) - u_R\right) dr}{\frac{R^2}{2}\left[J_1(\lambda_m R)\right]^2}$$

p(r) = r

California State University Northridge 11

Review Radial Solution IV

- Constant initial condition $u_0(r)$ = U

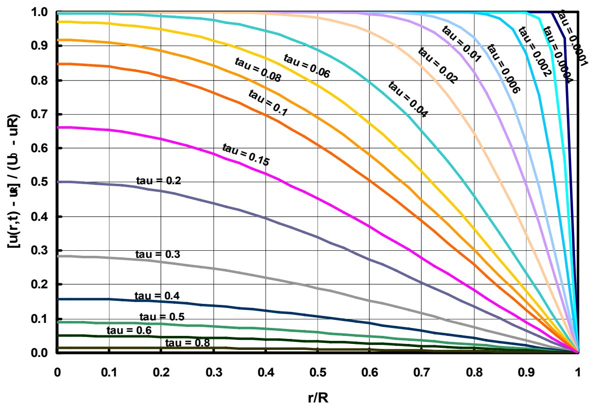
$$C_m = \frac{\int_0^R r J_0(\lambda_m r)\left(U - u_R\right) dr}{\frac{R^2}{2}\left[J_1(\lambda_m R)\right]^2} = \frac{2\left(U - u_R\right)}{\lambda_m R J_1(\lambda_m R)} = \frac{2\left(U - u_R\right)}{\alpha_{m0} J_1(\alpha_{m0})}$$

$$u(r,t) = \sum_{m=1}^{\infty} \frac{2\left(U - u_R\right)}{\alpha_{m0} J_1(\alpha_{m0})} e^{-\alpha_{m0}^2 \frac{\alpha t}{R^2}} J_0\left(\alpha_{m0} \frac{r}{R}\right) + u_R$$

$$\boxed{\frac{u - u_R}{U - u_R} = f\left(\frac{\alpha t}{R^2}, \frac{r}{R}\right)}$$

California State University Northridge 12

Radial Diffusion



Review Diffusion Equation

Rectangular	Radial
$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$	$\frac{\partial u}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r}$
$0 \le x \le L \quad t \ge 0$	$0 \le r \le R \quad t \ge 0$
$u(0,t) = 0, u_A$	$u(R,t) = 0, u_R$
$u(L,t) = 0, u_B$	$u(r,0) = u_0(r)$
$u(x,0) = u_0(x)$	$u(0,t)$ *finite*

Review Separation of Variables

Rectangular	Radial
$-\lambda^2 = \frac{1}{\alpha T(t)}\frac{\partial T(t)}{\partial t}$	$-\lambda^2 = \frac{1}{\alpha T(t)}\frac{\partial T(t)}{\partial t} =$
$= \frac{1}{\alpha X(x)}\frac{\partial^2 X(x)}{\partial x^2}$	$\frac{1}{\alpha r P(r)}\frac{\partial}{\partial r}\left(r\frac{\partial P(r)}{\partial r}\right)$
$T(t) = Ae^{-\lambda^2 \alpha t}$	$T(t) = Ae^{-\lambda^2 \alpha t}$

Review Zero Boundaries

Rectangular	Radial
$X(x) =$	$P(r) =$
$B\sin(\lambda x) + C\cos(\lambda x)$	$BJ_0(\lambda r) + CY_0(\lambda r)$
$C = 0$ *for* $u(0,t) = 0$	$C = 0$ *finite* $u(0,t) = 0$
$\lambda = \frac{n\pi}{L}$ *for* $u(L,t) = 0$	$\lambda = \frac{\alpha_{n0}}{R}$ *for* $u(R,t) = 0$
$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$	$P_n(r) = B_n J_0\left(\alpha_{n0}\frac{r}{R}\right)$

Review General Solution

Rectangular	Radial
$u(x,t) =$	$u(r,t) =$
$\sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \alpha t}\sin(\lambda_n x)$	$\sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \alpha t} J_0(\lambda_n r)$
$\lambda_n = n\pi / L$	$\lambda_n = \alpha_{n0} / R$
$u(x,0) = u_0(x) =$	$u(r,0) = u_0(r)$
$\sum_{n=1}^{\infty} C_n \sin(\lambda_n x)$	$\sum_{n=1}^{\infty} C_n J_0(\lambda_n r)$

Review C_n Equations

Rectangular	Radial
$\frac{\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx} = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$\frac{\int_0^R u_0(r) r J_0\left(\alpha_{n0}\frac{r}{R}\right)dr}{\int_0^R r J_0^2\left(\alpha_{n0}\frac{r}{R}\right)dr} = \frac{\int_0^R u_0(r) r J_0\left(\alpha_{n0}\frac{r}{R}\right)dr}{\frac{R^2}{2}J_1^2(\alpha_{n0})}$

Laplace's Equation

- Used to express equilibrium fields of engineering variable like temperature, species concentration, electrostatic potential and ideal fluid flow
- Written in general coordinates as $\nabla^2 u = 0$

Cartesian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

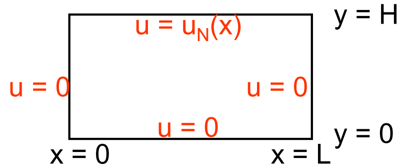
Cylindrical $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

Spherical $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$

California State University Northridge 19

Two-dimensional Problem

- Apply to rectangular region 0 ≤ x ≤ L and 0 ≤ y ≤ H
- Simple boundary conditions: u is zero at three surfaces and specified at other



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

California State University Northridge 20

Separation of Variables

- Assume u(x,y) = X(x)Y(y)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 [X(x)Y(y)]}{\partial x^2} = Y(y)\frac{\partial^2 X(x)}{\partial x^2} =$$

$$-\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 [X(x)Y(y)]}{\partial y^2} = -X(x)\frac{\partial^2 Y(y)}{\partial y^2}$$

- Divide by X(x)Y(y)

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\lambda^2$$

California State University Northridge 21

Solve ODEs to Get u(x,y)

- Two ODEs with known general solutions

$$\frac{d^2 X(x)}{dx^2} + \lambda^2 X(x) = 0 \qquad \frac{d^2 Y(y)}{dy^2} - \lambda^2 Y(y) = 0$$

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x)$$

$$Y(y) = C\sinh(\lambda y) + D\cosh(\lambda y)$$

$$u(x,y) = X(x)Y(y) = [A\sin(\lambda x) + B\cos(\lambda x)] \bullet [C\sinh(\lambda y) + D\cosh(\lambda y)]$$

California State University Northridge 22

Boundaries at x = 0 and x = L

- Since u(x,y) = X(x)Y(y), we must have X(0) = X(L) = 0 for u(0,y) = u(L,y) = 0
- Since X(x) = 0 at x = 0 and x = L, X(x) is the solution to a Sturm-Liouville problem
- X(x) = Asin(λx) + Bcos(λx)
- X(0) = 0 = Asin(0) + Bcos(0) => **B = 0**
- X(L) = 0 = Asin(λL)
- Must have **λL = nπ** (n an integer)
- Eigenfunction is sin(nπx/L)

California State University Northridge 23

Solution without u(x,H) = u_N(x)

- Boundary condition at y = 0, u(x,0) = 0, gives Y(0) = 0
- Y(0) = 0 = Csinh(0) + Dcosh(0) = D = 0
- With B = 0 and D = 0, solution is u(x,y) = Asin(λx) Csinh(λy) with λ = nπ/L
- General solution is sum of each eigenfunction multiplied by a constant

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n = \frac{n\pi}{L}$$

California State University Northridge 24

Separation Constant: λ^2 or $-\lambda^2$

- Have two choices in separation

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\lambda^2$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \lambda^2$$

- The first choice gives X(x) as sines and cosines; the second choice gives Y(y) as sines and cosines
  - Make choice to get sines and cosines in coordinate with two zero boundaries

California State University Northridge 25

Why Use sinh?

- Two equivalent solutions for the equation $d^2y/dx^2 - \lambda^2 y = 0$
  - $y = A \sinh \lambda x + B \cosh \lambda x$
  - $y = C e^{\lambda x} + D e^{-\lambda x}$
- Show equivalence from definitions of $\sinh z = (e^z - e^{-z})/2$; $\cosh z = (e^z + e^{-z})/2$
  - $C = (A + B)/2$ and $D = (B - A)/2$
- For y(0) = 0, sinh and cosh solution gives only one function, sinh

California State University Northridge 26

Boundary at y = H

- Multiply by sin(mπx/L) and integrate

$$u_N(x) = u(x,H) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)$$

- Use eigenfunction expansion to get boundary condition at y = H

$$\int_0^L u_N(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{m\pi H}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{m\pi H}{L}\right)\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = C_m \sinh\left(\frac{m\pi H}{L}\right)\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx$$

California State University Northridge 27

Orthogonal Eigenfunctions

- Solve for C_m

$$C_m = \frac{\int_0^L u_N(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\sinh\left(\frac{m\pi H}{L}\right)\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx} = \frac{\int_0^L u_N(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\sinh\left(\frac{m\pi H}{L}\right)\frac{L}{2}}$$

- Uses orthogonality for eigenfunctions

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \delta_{mn}\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{L\delta_{mn}}{2}$$

California State University Northridge 28

Boundary Condition Gives C_m

- Simplest case is $u_N(x) = U$, a constant

$$C_m = \frac{\int_0^L u_N(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\sinh\left(\frac{m\pi H}{L}\right)\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx} = \frac{\int_0^L U\sin\left(\frac{m\pi x}{L}\right)dx}{\sinh\left(\frac{m\pi H}{L}\right)\frac{L}{2}} = \frac{\frac{UL}{m\pi}(1-\cos m\pi)}{\sinh\left(\frac{m\pi H}{L}\right)\frac{L}{2}}$$

- 1 – cos(mπ) = 1 – (1) = 0 for even n and 1 – (–1) = 2 for odd n
- Write series using m = (2n+1) to get only odd terms

California State University Northridge 29

Constant Boundary Condition

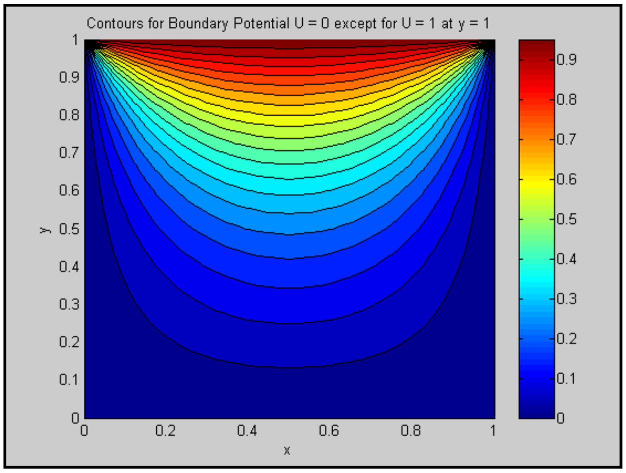
- Solution for u(x,y) is

$$u(x,y) = \frac{4U}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

C_m

- u/U = depends on x/L, y/L and H/L
- Plot on next page for H/L = 1

California State University Northridge 30



Similar to Diffusion Equation

Diffusion	Laplace
$\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$
$0 \le x \le L \quad t \ge 0$	$0 \le x \le L \quad 0 \le y \le H$
$u(0,t) = 0$	$u(0,y) = 0$
$u(L,t) = 0$	$u(L,y) = 0$
$u(x,t) = u_0(x)$	$u(x,H) = u_N(x)$
open boundary in t	$u(x,0) = 0$

Northridge 32

Separation of Variables

Diffusion	Laplace
$-\lambda^2 = \frac{1}{\alpha T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$	$-\lambda^2 = -\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$
$T(t) = Ae^{-\lambda^2 \alpha t}$	$Y(y) = A\sinh(\lambda y) + D\cosh(\lambda y)$

Northridge 33

x = 0,L Boundary Conditions

Diffusion	Laplace
$X(x) = B\sin(\lambda x) + C\cos(\lambda x)$	$X(x) = B\sin(\lambda x) + C\cos(\lambda x)$
$C = 0$ *for* $u(0,t) = 0$	$C = 0$ *for* $u(0,y) = 0$
$\lambda = \frac{n\pi}{L}$ *for* $u(L,t) = 0$	$\lambda = \frac{n\pi}{L}$ *for* $u(L,y) = 0$
$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$	$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$

Northridge 34

y = 0 Boundary Condition

Diffusion	Laplace
No equivalent condition because of open boundary in time	$Y(0) = A\sinh(\lambda 0) + D\cosh(\lambda 0) = 0$ *Must have* $D = 0$ $Y_n(y) = A_n \sinh\left(\frac{n\pi y}{L}\right)$

Northridge 35

General Solution; Fitted Condition

Diffusion	Laplace
$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \alpha t}\sin(\lambda_n x)$	$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\lambda_n y)\sin(\lambda_n x)$
$\lambda_n = n\pi/L$	$\lambda_n = n\pi/L$
$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x)$	$u(x,H) = u_N(x) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r)\sinh(\lambda_n H)$

Northridge 36

Eigenvalue Expansion for C_n

Diffusion	Laplace
$\dfrac{\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx} = \dfrac{2}{L}\int_0^L u_0(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$\dfrac{\int_0^L u_N(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{n\pi H}{L}\right)\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx} = \dfrac{\frac{2}{L}\int_0^L u_N(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{n\pi H}{L}\right)}$

Gradients as Fluxes

- Laplace and diffusion equation are based on conservation of fluxes which are (negative) gradients of potential
- Laplace equation gives equilibrium where net flux from region should be zero
- Details at end of presentation show the net outflow is zero for the solution here

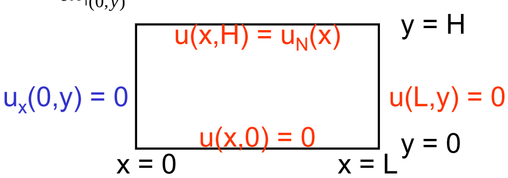
$$\text{Net outflow} = \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=0} dy - \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=L} dy + \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx - \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx = 0$$

Gradient Boundary Condition

- Consider the following problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 \le x \le L, \quad 0 \le y \le H$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,y)} = u(L,y) = u(x,0) = 0 \qquad u(x,H) = u_N(x)$$



Gradient Condition II

- Apply new boundary conditions to the previous separation of variables result

$$u(x,y) = X(x)Y(y) = [A\sin(\lambda x) + B\cos(\lambda x)] \bullet [C\sinh(\lambda y) + D\cosh(\lambda y)]$$

$$u_x = \frac{\partial u}{\partial x} = X'(x)Y(y) = \lambda[A\cos(\lambda x) - B\sin(\lambda x)] \bullet [C\sinh(\lambda y) + D\cosh(\lambda y)]$$

Boundaries at x = 0 and x = L

- For $u_x = 0$ at x = 0 and u = 0 at x = L, we have X'(0) = X(L) =0 so X(x) is the solution to a Sturm-Liouville problem
- X(x) = Asin(λx) + Bcos(λx)
- X'(x) = λAcos(λx) – λBsin(λx)
- X'(0) = 0 = λAcos(0) – λBsin(0): **A = 0**
- X(L) = 0 = Bcos(λL)
- Must have **λL = mπ/2** (m an odd integer; use m = (2n + 1)
- Eigenfunction is cos[(2n+1)πx/2L]

Solution without $u(x,H) = u_N(x)$

- Boundary condition at y = 0, u(x,0) = 0, gives Y(0) = 0
- Y(0) = 0 = Csinh(0) + Dcosh(0) = D = 0
- With A = 0 and D = 0, solution is u(x,y) = Bcos(λx) Csinh(λy) with λ = (n+1/2)π/L
- General solution is sum of each eigenfunction multiplied by a constant

$$u(x,y) = \sum_{n=0}^{\infty} C_n \cos(\lambda_n x)\sinh(\lambda_n y) \qquad \lambda_n = \frac{(2n+1)\pi}{2L}$$

Boundary at y = H

- Eigenfunction expansion

$$u_N(x) = u(x,H) = \sum_{n=0}^{\infty} C_n \cos(\lambda_n x)\sinh(\lambda_n H) \quad \lambda_n = \frac{(2n+1)\pi}{2L}$$

- Usual solution process gives

$$C_n = \frac{\int_0^L u_N(x)\cos(\lambda_n x)dx}{\sinh(\lambda_n H)\int_0^L \cos^2(\lambda_n x)dx} = \frac{\int_0^L u_N(x)\cos(\lambda_n x)dx}{\sinh(\lambda_n H)\frac{L}{2}}$$

Boundary Condition Gives C_m

- Simplest case is $u_N(x) = U$, a constant

$$C_n = \frac{\int_0^L u_N(x)\cos(\lambda_n x)dx}{\sinh(\lambda_n H)\frac{L}{2}} = \frac{\int_0^L U\cos(\lambda_n x)dx}{\sinh(\lambda_n H)\frac{L}{2}} = \frac{\frac{U}{\lambda_n}(\sin\lambda_n L - \sin 0)}{\sinh(\lambda_n H)\frac{L}{2}}$$

- $\sin \lambda_n L = \sin[(2n+1)\pi/2] = 1$ for even n and –1 for odd n which is $(-1)^n$

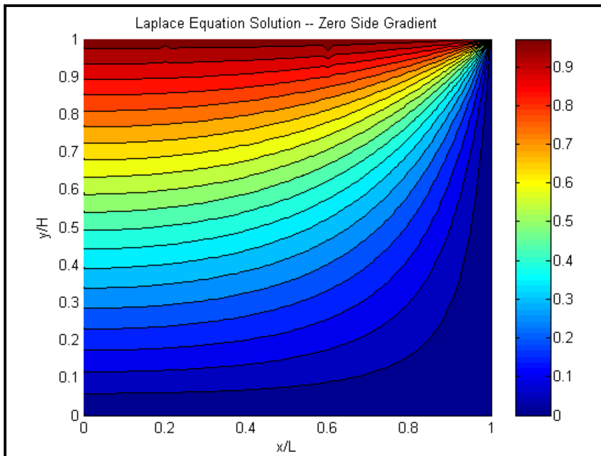
$$C_n = \frac{2U(-1)^n}{\lambda_n L\sinh(\lambda_n H)} = \frac{4U(-1)^n}{(2n+1)\pi\sinh\left(\frac{(2n+1)\pi H}{2L}\right)}$$

Constant Boundary Solution

- Substitute C_n and $\lambda_n = (2n+1)\pi/2L$ into general solution for u(x,y)

$$u(x,y) = \frac{4U}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos\left(\frac{(2n+1)\pi x}{2L}\right)\sinh\left(\frac{(2n+1)\pi y}{2L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{2L}\right)}$$

- u/U = depends on x/L, y/L and H/L
- Plot on next page for H/L = 1



Summary

- Review radial diffusion equation
  – Solution in Bessel functions
  – Eigenfunction expansion of Bessel functions to match initial condition
  – Can handle other boundary conditions
- Laplace equation also solved by separation of variables
  – Three zero boundaries and non nonzero boundary
  – Zero net flux into region for one case
  – Also did zero flux boundary condition

Details of Zero Net Flux

- For heat flux $q_x = -k\partial T/\partial x$ $q_y = -k\partial T/\partial y$
- Net heat flux leaving region is integrated heat flux at all four boundaries

$$Net\ output = \int_0^H k\left.\frac{\partial T}{\partial x}\right|_{x=0} dy - \int_0^H k\left.\frac{\partial T}{\partial x}\right|_{x=L} dy + \int_0^L k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx - \int_0^L k\left.\frac{\partial T}{\partial y}\right|_{y=H} dx$$

- Remaining charts evaluate net output
  – Take derivatives of solution for u(x,y) as T(x,y), evaluate at x = 0, x = L, y = 0, and y = H; integrate and add as shown above

x-Direction Gradients

$$u(x,y) = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

$$\frac{\partial u}{\partial x} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- Same inflow at x = 0 and x = L as expected by symmetry of solution
- cos(0) = 1 and cos[(2n+1)π] = –1 so

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -\frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

49

Integrated x-Direction Inflow

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -\frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

$$\int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=0} dy - \int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=L} dy = 2\int_0^H \left.\frac{\partial u}{\partial x}\right|_{x=0} dy = 2\int_0^H \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} dy$$

$$= \frac{8U}{\pi}\sum_{n=1}^{\infty}\frac{\left[\cosh\left(\frac{(2n+1)\pi y}{L}\right)\right]_0^H}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)} = \frac{8U}{\pi}\sum_{n=1}^{\infty}\frac{\cosh\left(\frac{(2n+1)\pi H}{L}\right)-1}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

50

y-Direction Gradients

$$u(x,y) = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\sinh\left(\frac{(2n+1)\pi y}{L}\right)}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

$$\frac{\partial u}{\partial y} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)\cosh\left(\frac{(2n+1)\pi y}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

- cosh(0) = 1 and tanh(x) = sinh(x) / cosh(x)

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=H} = \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)}{\tanh\left(\frac{(2n+1)\pi H}{L}\right)}$$

51

Net y-Direction Outflow

$$\int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx = \int_0^L \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} dx = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\left[-\cos\left(\frac{(2n+1)\pi x}{L}\right)\right]_0^L}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

$$\int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx = \int_0^L \frac{4U}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)}{\tanh\left(\frac{(2n+1)\pi H}{L}\right)} dx = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\left[-\cos\left(\frac{(2n+1)\pi x}{L}\right)\right]_0^L}{(2n+1)\tanh\left(\frac{(2n+1)\pi H}{L}\right)}$$

$$\int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx - \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{2}{(2n+1)}\left[\frac{1}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} - \frac{1}{\tanh\left(\frac{(2n+1)\pi H}{L}\right)}\right]$$

52

Total Net Outflow Is Zero

$$\int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx - \int_0^L \left.\frac{\partial u}{\partial y}\right|_{y=H} dx = \frac{8U}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\left[\frac{1}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)} - \frac{1}{\tanh\left(\frac{(2n+1)\pi H}{L}\right)}\right]$$

$$\int_0^H \left.\frac{\partial T}{\partial x}\right|_{x=0} dy - \int_0^H \left.\frac{\partial T}{\partial x}\right|_{x=L} dy = \frac{8U}{\pi}\sum_{n=1}^{\infty}\frac{\cosh\left(\frac{(2n+1)\pi H}{L}\right)-1}{(2n+1)\sinh\left(\frac{(2n+1)\pi H}{L}\right)}$$

Sum is zero

Equal and opposite

$$= \frac{8U}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\left[\frac{1}{\tanh\left(\frac{(2n+1)\pi H}{L}\right)} - \frac{1}{\sinh\left(\frac{(2n+1)\pi H}{L}\right)}\right]$$

53