The Diffusion Equation

Larry Caretto
Mechanical Engineering 501B

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Overview

- Review last week
- Diffusion equation
  - Physical meaning and derivation
  - Relation to Laplace equation
  - Solution by separation of variables
  - Sturm-Liouville orthogonal eigenfunction expansion for initial conditions
    - Only possible for homogenous boundary conditions
    - Treatment of nonhomogenous boundary conditions

Review Sturm-Liouville

- Homogenous equations for a ≤ x ≤ b

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)+\left[q(x)+\lambda p(x)\right]y=0$$

$$k_1 y(a)+k_2\left.\frac{dy}{dx}\right|_{x=a}=0 \qquad \ell_1 y(b)+\ell_2\left.\frac{dy}{dx}\right|_{x=b}=0$$

- Solutions, y_m are complete set of orthogonal eignenfunctions that can be used to expand any function $f(x)=\sum_{m=0}^{\infty} a_m y_m(x)$

Review Orthogonal Functions

- Defined as inner product integral with p(x) from Sturm-Liouville equation

$$\left(y_i, y_j\right)=\int_a^b y_i^*(x)y_j(x)p(x)dx=\left\|y_i\right\|^2\delta_{ij}$$

- Get coefficients in eigenfunction expansions

$$a_m=\frac{(y_m, f)}{\left(y_m, y_m\right)}=\frac{\int_a^b p(x)y_m(x)f(x)dx}{\int_a^b p(x)y_m(x)y_m(x)dx}$$

Review Fourier Series

- Based on periodic functions defined over –L < x < L

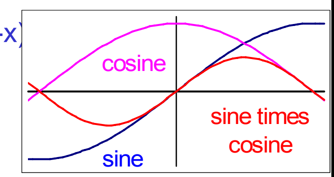
$$f(x)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Review Even/Odd Functions

- Odd function: f(-x) = -f(x) (like sine)
- Even function: g(-x) = g(x) (like cosine)



- Product of odd times even is odd
- For odd f(x)

$$\int_{-L}^{L} f(x)dx=0$$

- For even g(x)

$$\int_{-L}^{L} g(x)dx=2\int_0^L g(x)dx$$

Review Half-Interval Series

- For even functions, there are no sine terms in the Fourier series
- For odd functions there are no cosine terms in the Fourier series
- In these cases can use integrals from 0 to L for the coefficients
- Can also use equations from 0 to L for all functions, but get periodic behavior outside this region

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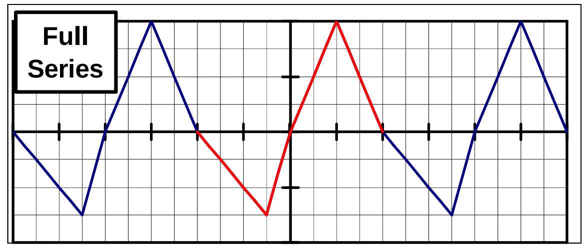
Review Half-Interval Series II

- Look at function below for L = 2
- Full series gives periodic results



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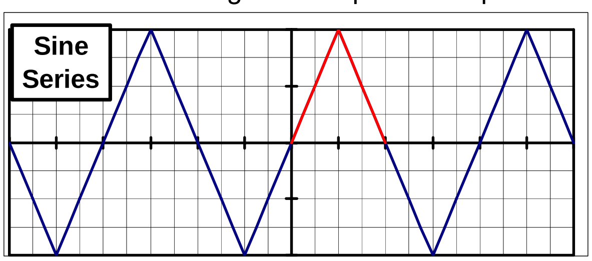
Review Half-Interval Series III

- Expanding function for 0 ≤ x ≤ L = 2 with a sine series gives odd periodic repetition



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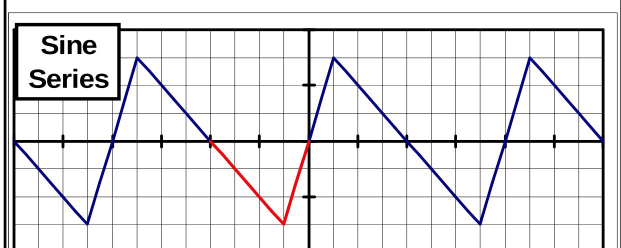
Review Half-Interval Series IV

- Expanding function for –L ≤ x ≤ 0 with a sine series gives odd periodic repetition



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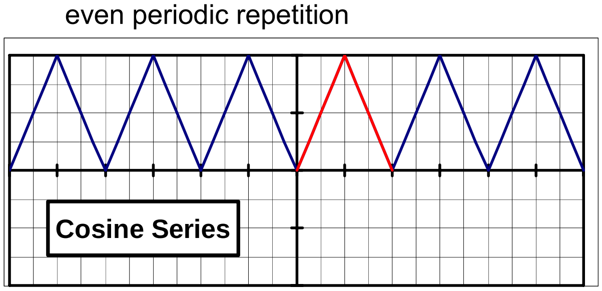
Review Half-Interval Series V

- Cosine series for 0 ≤ x ≤ L = 2 with gives even periodic repetition



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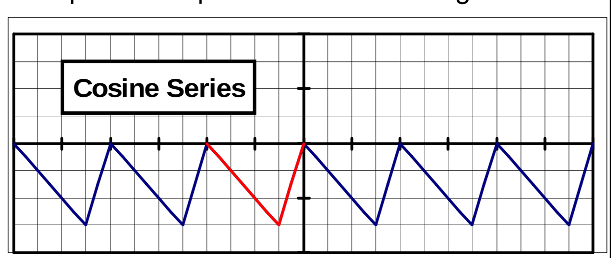
Review Half-Interval Series VI

- Cosine series for –L ≤ x ≤ 0 gives even periodic repetition of different region



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Diffusion and Laplace Equations

- Partial differential equations related to conservation principles of fluxes governed by potentials
  - Heat transfer from temperature gradient
  - Mass diffusion from concentration gradient
  - Current from electrostatic potential
  - Magnetic fluxes
  - Ideal fluid flow from velocity potential
- Laplace is steady state case of diffusion

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Derivation Basics

- Consider a flux, f, that is a gradient of a potential, u, f = –ρα∂u/∂x
- The net outflow is $f|_{x+dx}\Delta A - f|_x\Delta A$
- Dividing by ΔV = ΔAΔx gives the net outflow per unit volume = $(f|_{x+dx} - f|_x)/\Delta x$ which is ∂f/∂x as Δx approaches zero
- Substituting the potential definition gives the net outflow (per unit volume) as ∂[–ρα∂u/∂x]/∂x

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Derivation Basics II

- A conserved quantity has zero net outflow for steady processes with no source terms
  - This gives ∂[-ρα∂u/∂x]/∂x = 0 or ∂[-ρα∂u/∂x]/∂x + ∂[-ρα∂u/∂y]/∂y + ∂[-ρα∂u/∂z]/∂z = 0; constant properties ρα gives Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\nabla^2 u=0$$

- Can be two-dimensional
- Changes with coordinate systems

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Derivation Basics III

- For transient processes, a net outflow causes a decrease in ρu
  - This gives ∂ρu/∂t = ∂[ρα∂u/∂x]/∂x + ∂[ρα∂u/∂y]/∂y + ∂[ρα∂u/∂z]/∂z
  - For constant properties ρ and α we have the diffusion equation

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\alpha\nabla^2 u$$

- Can have one or two space dimensions
- Changes with coordinate systems

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Derivation Basics IV

- Diffusion equation gives transient process for systems where steady state is Laplace's equation

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\alpha\nabla^2 u \qquad \nabla^2 u=0$$

transient steady-state

- Can have different space dimensions
- α, called the diffusion coefficient or diffusivity, has dimensions of (length)^2/time

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Multidimensional Equations

- General diffusion equation for three dimensions $\frac{\partial u}{\partial t}=\alpha\nabla^2 u$

$$\textit{Cartesian} \quad \nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$\textit{Cylindrical} \quad \nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$$

$$\textit{Sphere} \quad \nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}$$

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Variable Properties

- Included in outer space derivative

$$\frac{\partial \rho u}{\partial t} = \nabla \cdot (\rho\alpha \nabla u) = div(\rho\alpha \; grad \; u)$$

$$Cartesian \quad \frac{\partial \rho u}{\partial t} = \frac{\partial}{\partial x}\rho\alpha\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\rho\alpha\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\rho\alpha\frac{\partial u}{\partial z}$$

$$Cylindrical \quad \frac{\partial \rho u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\rho\alpha r\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\rho\alpha\frac{\partial u}{\partial \theta} + \frac{\partial}{\partial z}\rho\alpha\frac{\partial u}{\partial z}$$

$$Sphere \quad \frac{\partial \rho u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\rho\alpha r^2\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial}{\partial \theta}\rho\alpha\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\rho\alpha\sin\phi\frac{\partial u}{\partial \phi}$$

Diffusion Equation Solutions

- Governs heat conduction and species diffusion for $t \geq 0$ and $0 \leq x \leq x_{max}$
  - u(x,t) is temperature, species concentration
  - Initial condition $u(x,0) = u_0(x)$
  - Boundaries $u(0,t) = u_L(t)$; $u(x_{max},t) = u_R(t)$
- Diffusivity, α, is material property $(length)^2/(time)$

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

Separation of Variables

- Assume u(x,t) = X(x)T(t)

$$\frac{\partial u}{\partial t} = \frac{\partial[X(x)T(t)]}{\partial t} = X(x)\frac{\partial T(t)}{\partial t} =$$

$$\alpha\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2[X(x)T(t)]}{\partial x^2} = \alpha T(t)\frac{\partial^2 X(x)}{\partial x^2}$$

- Divide by αX(x)T(t)

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}$$

Result is function of t equal to function of x

Separation of Variables Works

- Assumed solution, u(x,t) = X(x)T(t), gives a function of x equal a function of t
- Since x and t are independent, both sides must equal a constant for this to be true

$$\frac{1}{\alpha}\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\lambda^2$$

- Choose negative real constant, $-\lambda^2$, to simplify later solution of resulting ordinary differential equations (ODE)

Solve ODEs to Get u(x,t)

- Have simple ODEs with known general solutions

$$\frac{dT(t)}{dt} = -\lambda^2\alpha T(t) \qquad T(t) = Ae^{-\lambda^2\alpha t}$$

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X(x) = 0 \qquad X(x) = B\sin(\lambda x) + C\cos(\lambda x)$$

$$u(x,t) = T(t)X(x) = Ae^{-\lambda^2\alpha t}[B\sin(\lambda x) + C\cos(\lambda x)]$$

$$= e^{-\lambda^2\alpha t}[C_1\sin(\lambda x) + C_2\cos(\lambda x)]$$

Boundary Conditions

- Look at case where $u(0,t) = u(x_{max},t) = 0$
- Since $u(0,t) = T(t)X(0)$ and $u(x_{max},t) = T(t)X(x_{max})$ we can only satisfy these boundary conditions if $X(0) = X(x_{max}) = 0$
  - Because the ODE for X(x) and its boundary conditions are homogenous, X(x) is the solution to a Sturm-Liouville problem
- X(x) = Bsin(λx) + Ccos(λx)
  - How do we get X(0) = 0?
  - X(0) = 0 = Bsin(0) + Ccos(0) = C

Boundary Conditions II

- We must have C = 0 for X(0) = 0
  - Thus X(x) = Bsin(λx)
- How do we get second boundary condition that $X(x_{max}) = 0$?
- $X(x_{max}) = 0 = B\sin(\lambda x_{max})$
- This is true if B = 0 (which gives X always zero) or if $\sin(\lambda x_{max}) = 0$
  - $\sin(\lambda x_{max}) = 0$ if $\lambda x_{max} = n\pi$ (n an integer)
- Eigenfunction is $X(x) = B_n\sin(n\pi x/x_{max})$

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Eigenfunction Solutions

- Original solution is u(x,t) = T(t)X(x)
- We found an infinite set of eigen-functions for $X_n(x) = B_n\sin(n\pi x/x_{max})$
- We also found $T(t) = Ae^{-\lambda^2\alpha t}$
- Have an infinite set of solutions where $\lambda_n = n\pi x/x_{max}$:

$$u_n(x,t) = Ae^{-\lambda_n^2\alpha t}B_n\sin(\lambda_n t)$$

- General solution for u(x,t) is sum of all individual eigenfunction solutions

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Initial Condition

- In general solution (sum of all eigensolutions) write the product of two constants AB_n as C_n

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2\alpha t}\sin(\lambda_n x) \qquad \lambda_n = \frac{n\pi}{x_{max}}$$

- Setting t = 0 gives eigenfunction expansion for the initial condition

$$u_0(x) = u(x,0) = \sum_{n=1}^{\infty} C_n e^0 \sin\left(\frac{n\pi x}{x_{max}}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{x_{max}}\right)$$

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Finding C_n

- We already know the answer
- This is a Sturm-Liouville problem so the eigenfunctions are a complete, orthogonal set
  - $p(x) = 1$ and $X_n(x) = \sin(n\pi x/x_{max})$

$$C_n = \frac{(X_n, u_0)}{(X_n, X_n)} = \frac{\int_a^b p(x)X_n(x)u_0(x)dx}{\int_a^b p(x)X_n(x)X_n(x)dx}$$

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Derivation of C_n Equation

- We can derive the C_n equation on the previous page
  - Since the X(x) is the solution of a Sturm-Liouville problem we have an infinite set of orthogonal eigenfunctions
  - Multiply basic equation for the initial condition by $\sin(m\pi x/x_{max})$ and integrate from 0 to x_{max}

$$\int_0^{x_{max}} \sin\left(\frac{m\pi x}{x_{max}}\right)\left[u_0(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{x_{max}}\right)\right]dx$$

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Derivation of C_n Equation II

- Consider each side separately

$$\int_0^{x_{max}} u_0(x)\sin\left(\frac{m\pi x}{x_{max}}\right)dx = \int_0^{x_{max}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{x_{max}}\right)\sin\left(\frac{m\pi x}{x_{max}}\right)dx$$

$$= \sum_{n=1}^{\infty} C_n \int_0^{x_{max}} \sin\left(\frac{m\pi x}{x_{max}}\right)\sin\left(\frac{n\pi x}{x_{max}}\right)dx = C_m \int_0^{x_{max}} \sin^2\left(\frac{m\pi x}{x_{max}}\right)dx$$

- Uses orthogonality for eigenfunctions

$$\int_0^{x_{max}} \sin\left(\frac{m\pi x}{x_{max}}\right)\sin\left(\frac{n\pi x}{x_{max}}\right)dx = \delta_{mn}\int_0^{x_{max}} \sin^2\left(\frac{m\pi x}{x_{max}}\right)dx = \frac{x_{max}\delta_{mn}}{2}$$

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Initial Condition Gives C_m

- Solve previous result for C_m

General result for any initial condition

$$C_m = \frac{\int_0^{x_{max}} u_0(x)\sin\left(\frac{m\pi x}{x_{max}}\right)dx}{\int_0^{x_{max}} \sin^2\left(\frac{m\pi x}{x_{max}}\right)dx} = \boxed{\frac{2}{x_{max}}\int_0^{x_{max}} u_0(x)\sin\left(\frac{m\pi x}{x_{max}}\right)dx}$$

- Simplest case is $u_0(x) = U$, a constant

$$C_m = \frac{2U}{x_{max}}\int_0^{x_{max}} \sin\left(\frac{m\pi x}{x_{max}}\right)dx = \frac{2U}{x_{max}}\left[-\frac{x_{max}}{m\pi}\cos\left(\frac{m\pi x}{x_{max}}\right)\right]_0^{x_{max}} = \frac{2U}{m\pi}\left[1-\cos(m\pi)\right]$$

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Constant Initial Condition

- $1 - \cos(m\pi) = 1 - (1) = 0$ for even m and $1 - (-1) = 2$ for odd m
  - $C_m = 0$ for even m and $4U/m\pi$ for odd m
- Write series using $m = (2n+1)$ to get only odd terms with increment of one

$$u(x,t) = \frac{4U}{\pi}\sum_{n=0}^{\infty}\frac{e^{-\frac{(2n+1)^2\pi^2\alpha t}{x_{max}^2}}\sin\left(\frac{(2n+1)\pi x}{x_{max}}\right)}{2n+1}$$

- u/U = depends on x/x_{max} and $\alpha t/x_{max}^2$

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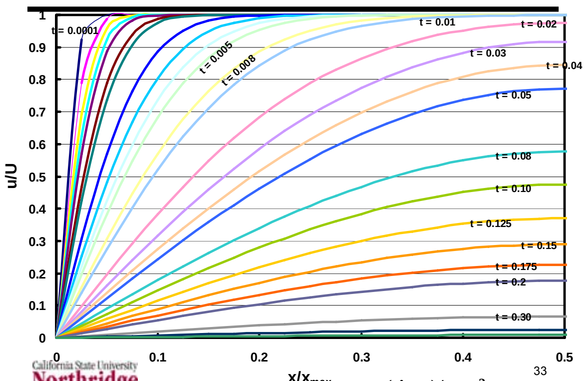
Diffusion Equation Solution



$t = \alpha(\text{time})/x_{max}^2$

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Nonzero Boundaries

- Sturm-Liouville eigenfunction expansion requires zero boundary conditions
- For nonzero boundaries, $u(0,t) = u_L$ and $u(x_{max},t) = u_R$, split solution into two functions $u(x,t) = v(x,t) + w(x)$
- Have to satisfy original PDE: $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial(v+w)}{\partial t} = \alpha\frac{\partial^2(v+w)}{\partial x^2} \Rightarrow \frac{\partial v}{\partial t} = \alpha\frac{\partial^2 v}{\partial x^2} + \alpha\frac{\partial^2 w}{\partial x^2}$$

Set this to zero

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Nonzero Boundaries II

- Have two differential equations

$$\frac{\partial v}{\partial t} = \alpha\frac{\partial^2 v}{\partial x^2} \qquad and \qquad \frac{\partial^2 w}{\partial x^2} = \frac{d^2 w}{dx^2} = 0$$

- To satisfy $u(0,t) = u_L$ and $u(x_{max},t) = u_R$, set $v(0,t) = v(x_{max},t) = 0$, $w(0) = u_L$ and $w(x_{max}) = u_R$
  - $u(0,t) = v(0,t) + w(0) = 0 + u_R = u_R$
  - $u(x_{max},t) = v(x_{max},t) + w(x_{max}) = 0 + u_L = u_L$
  - Differential equation and boundary conditions on u are satisfied

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Nonzero Boundaries Summary

- $u(x,t) = v(x,t) + w(x)$ satisfies the PDE

w = w(x) only 0 $d^2w/dx^2 = 0$

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial v}{\partial t} + \cancel{\frac{\partial w}{\partial t}} = \alpha\frac{\partial^2 v}{\partial x^2} + \cancel{\alpha\frac{\partial^2 w}{\partial x^2}}$$

Equal by v definition

- The boundary conditions on $v(x,t)$ and $w(x)$ satisfy the conditions on $u(x,t)$
  - $u(0,t) = v(0,t) + w(0) = 0 + u_L = u_L$
  - $u(x_{max},t) = v(x_{max},t) + w(x_{max}) = 0 + u_R = u_R$
- Initial condition: $v(x,0) = u(x,0) - w(x)$

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w(x) Is Steady Solution

- Solution for $d^2w/dx^2 = 0$ with $w(0) = u_L$ and $w(x_{max}) = u_R$ is $w = u_L + (u_R - u_L)x/x_{max}$
- Solution for v is same as previous solution for u with $u(0,t) = u(x_{max},t) = 0$

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 \alpha t} \sin(\lambda_n x) \qquad \lambda_n = \frac{n\pi}{x_{max}}$$

- Initial condition for $v(x,0) = u(x,0) - w(x) = u_0(x) - [\, u_L + (u_R - u_L)x/x_{max} \,]$

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Nonzero Boundaries IV

$$u(x,t) = \underbrace{\sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right)}_{v(x,t)} + \underbrace{u_L + \frac{u_r - u_L}{x_{max}} x}_{w(x)}$$

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{x_{max}}\right) + u_L + \frac{u_r - u_L}{x_{max}} x$$

- Use eigenfunction expansion for $v(x,0) = u_0(x) - u_L - (u_R - u_L)x/x_{max}$
  - $v(x,0)$ is a Sturm-Liouville solution

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Nonzero Boundaries V

- Same eigenfunction expansion used for zero boundary problem gives

$$C_m = \frac{2}{x_{max}} \int_0^{x_{max}} \underbrace{\left[u_0(x) - u_L - \frac{u_R - u_L}{x_{max}} x \right]}_{=v(x,0)} \sin\left(\frac{m\pi x}{x_{max}}\right) dx$$

- After C_n is found, solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) + u_L + \frac{u_r - u_L}{x_{max}} x$$

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Nonzero Boundary Example

- Solution from previous slide

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) + u_L + \frac{u_r - u_L}{x_{max}} x$$

$$C_n = \frac{2}{x_{max}} \int_0^{x_{max}} \left[u_0(x) - u_L - \frac{u_R - u_L}{x_{max}} x \right] \sin\left(\frac{n\pi x}{x_{max}}\right) dx$$

- The solution for $u_0(x) = U$, a constant, is found from equations above by setting $u_0(x) = U$ and integrating to get C_n

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Nonzero Boundary Result

- Solution details (not covered in lecture) for $u_0(x) = U_0$ are on slides 46-52
  - Final form shows that $(u(x,t) - u_L)/(U_0 - u_L)$ is a function of x/x_{max}, $\alpha t/L^2$ and $(u_R - u_L)/(U_0 - u_L)$

$$\frac{u(x,t) - u_L}{U_0 - u_{:L}} = \frac{u_R - u_L}{U_0 - u_{:L}} \left[\frac{x}{x_{max}} - \frac{2}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) \right]$$

$$+ \frac{2}{\pi} \left(2 - \frac{u_R - u_L}{U_0 - u_{:L}} \right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right)$$

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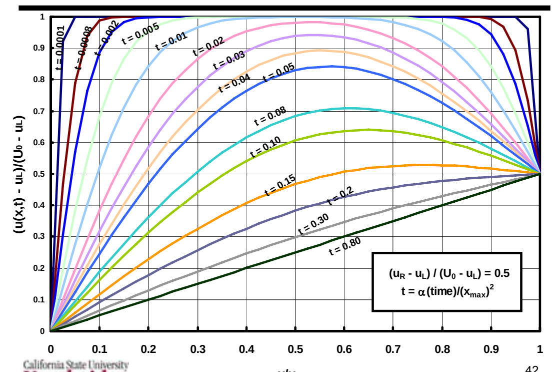
Non-zero Boundary Plot



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Other Boundary Conditions

- Look at gradient and mixed boundary conditions
- Show how making differential equation dimensionless can lead to Sturm-Liouville problem in initial formulation
- Will consider these next class and on homework for next week
- Can give cosine eigenfunctions and irregular eigenvalues

Diffusion Equation Summary

- Create Sturm-Lioville problem for non-zero spatial boundary conditions
  - Define u(x,t) = v(x,t) + w(x)
  - Use u – u_{ref} in original equation
- Solve by separation of variables
  - Time solution will be exponential
- Apply boundary conditions to determine eigenvalues (also gets constants in solution)

Diffusion Equation Summary II

- Write solution as sum of all possible eigenfunctions with individual constants
- Use eigenfunction expansion to match initial conditions
  - If a solution for u(x,t) = v(x,t) + w(x) is used the eigenfunction expansion must be for u_0(x) – w(x)
- Solution is sum of all eigenfunctions with constants determined from matching initial conditions.

Nonzero Boundary Example

- Constant initial condition, $u_0(x) = U_0$

$$C_m = \frac{2}{x_{\max}} \int_0^{x_{\max}} \left[U_0 - u_L - \frac{u_R - u_L}{x_{\max}} x \right] \sin\left(\frac{m\pi x}{x_{\max}} \right) dx = I_1 - I_2$$

$$I_1 = \frac{2(U_0 - u_L)}{x_{\max}} \int_0^{x_{\max}} \sin\left(\frac{m\pi x}{x_{\max}} \right) dx \quad I_2 = \frac{2(u_R - u_L)}{x_{\max}^2} \int_0^{x_{\max}} x \sin\left(\frac{m\pi x}{x_{\max}} \right) dx$$

$$I_1 = \frac{2(U_0 - u_L)}{x_{\max}} \frac{x_{\max}}{m\pi} \left[-\cos\left(\frac{m\pi x}{x_{\max}} \right) \right]_0^{x_{\max}}$$

$$I_2 = \frac{2(u_R - u_L)}{x_{\max}^2} \left[\frac{x_{\max}^2}{m^2\pi^2} \sin\left(\frac{m\pi x}{x_{\max}} \right) - \frac{x_{\max}}{m\pi} x \cos\left(\frac{m\pi x}{x_{\max}} \right) \right]_0^{x_{\max}}$$

Nonzero Boundary Example II

- First integral

$$I_1 = \frac{2(U_0 - u_L)}{x_{\max}} \frac{x_{\max}}{m\pi} \left[-\cos\left(\frac{m\pi x}{x_{\max}} \right) \right]_0^{x_{\max}} = \frac{2(U_0 - u_L)}{x_{\max}} \frac{x_{\max}}{m\pi} \left[-\cos(m\pi) + 1 \right]$$

- 1 – cos(mπ) = 0 for even m and 2 for odd m

$$I_1 = \begin{cases} \dfrac{4(U_0 - u_L)}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

Second Integral

$$I_2 = \frac{2(u_R - u_L)}{x_{\max}^2} \left[\frac{x_{\max}^2}{m^2\pi^2} \sin\left(\frac{m\pi x}{x_{\max}} \right) - \frac{x_{\max}}{m\pi} x \cos\left(\frac{m\pi x}{x_{\max}} \right) \right]_0^{x_{\max}}$$

$$= \frac{2(u_R - u_L)}{x_{\max}^2} \left[\frac{x_{\max}^2}{m^2\pi^2} (\sin m\pi - \sin 0) \right] -$$

$$\frac{2(u_R - u_L)}{x_{\max}^2} \left[\frac{x_{\max}}{m\pi} x_{\max} \cos(m\pi) - \frac{x_{\max}}{m\pi} (0) \cos(0) \right]$$

$$= \frac{2(u_R - u_L)}{x_{\max}^2} [0] - \frac{2(u_R - u_L)}{x_{\max}^2} \frac{x_{\max}}{m\pi} x_{\max} \cos(m\pi)$$

Because cos(mπ) = 1 when m is even and –1 when m is odd

$$I_2 = \begin{cases} \dfrac{2(u_R - u_L)}{m\pi} & m \text{ odd} \\ -\dfrac{2(u_R - u_L)}{m\pi} & m \text{ even} \end{cases}$$

Nonzero Boundary Example III

$$C_m = I_1 - I_2 = \frac{2(U_0 - u_L)}{m\pi}[1-\cos(m\pi)] - \left[-\frac{2(u_R - u_L)}{m\pi}\cos m\pi\right]$$

$$C_m = \begin{cases} \dfrac{4(U_0 - u_L)}{m\pi} - \dfrac{2(u_R - u_L)}{m\pi} & m\ odd \\ \dfrac{2(u_R - u_L)}{m\pi} & m\ even \end{cases}$$

$$\boxed{1-\cos m\pi = \begin{cases} 0 & m\ even \\ 2 & m\ odd \end{cases}}$$

$$\frac{C_m}{(U_0 - u_L)} = \begin{cases} \dfrac{4}{m\pi} - \dfrac{2(u_R - u_L)}{m\pi(U_0 - u_L)} & m\ odd \\ \dfrac{2(u_R - u_L)}{m\pi(U_0 - u_L)} & m\ even \end{cases}$$

California State University Northridge 49

Nonzero Boundary Example IV

- We want to get dimensionless form to plot results with significant parameters
  - Show $(u(x,t) - u_L)/(U_0 - u_L)$ to be function of x/x_{max}, $\alpha t/x_{max}^2$ and $(u_R - u_L)/(U_0 - u_L)$
    - Alternative analysis if $u_L = U_0$ uses $(u(x,t) - u_L)/(u_R - u_L)$
    - At least one analysis is possible because $u(x,t) = U_0$ if $u_L = u_R = U_0$
- Use equation from last chart for $C_n/(U_0 - u_L)$ in general solution

California State University Northridge 50

Nonzero Boundary Example V

- General solution for any $u_0(x)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) + u_L + \frac{u_R - u_L}{x_{max}} x$$

- Subtract u_L from both sides and divide by $U_0 - u_L$

$$\frac{u(x,t) - u_L}{U_0 - u_L} = \sum_{n=1}^{\infty} \frac{C_n}{U_0 - u_L} e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) + \frac{u_R - u_L}{U_0 - u_L}\frac{x}{x_{max}}$$

- Write separate sums with odd and even n equations for $C_n/(U_0 - u_L)$ from previous result

California State University Northridge 51

Nonzero Boundary Example VI

$$\frac{u(x,t) - u_L}{U_0 - u_L} = \sum_{n=1,3,5,\ldots}^{\infty} \left[\frac{4}{n\pi} - \frac{2(u_R - u_L)}{n\pi(U_0 - u_L)}\right] e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right)$$

$$\sum_{n=2,4,6,\ldots}^{\infty} \left[\frac{2(u_R - u_L)}{n\pi(U_0 - u_L)}\right] e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) + \frac{u_R - u_L}{U_0 - u_L}\frac{x}{x_{max}}$$

$$\boxed{\begin{aligned} \frac{u(x,t) - u_L}{U_0 - u_{:L}} &= \frac{u_R - u_L}{U_0 - u_{:L}}\left[\frac{x}{x_{max}} - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right)\right] \\ &+ \frac{2}{\pi}\left(2 - \frac{u_R - u_L}{U_0 - u_{:L}}\right)\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{x_{max}}\right)^2 \alpha t} \sin\left(\frac{n\pi x}{x_{max}}\right) \end{aligned}}$$

California State University Northridge 52