Fourier Series

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Seminar in Engineering Analysis

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Outline

- Review last class
- Fourier series as expansions in periodic functions
  - Comparison to eigenfunction expansions
- Odd and even functions
- Periodic extensions of non-periodic functions
- Complex Fourier series

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Review Last Lecture

- Discussed Sturm-Liouville Problem
- Solutions are a set of orthogonal eigenfunctions, $y_m(x)$ that can be used to express other functions, f(x)
- p(x) is weight function
- Have to compute a_m

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$$

$$a_m = \frac{(y_m, f)}{(y_m, y_m)} = \frac{\int_a^b p(x) y_m(x) f(x) dx}{\int_a^b p(x) y_m(x) y_m(x) dx}$$

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Review Orthogonal Functions

- Defined in terms of inner product
- Norm of two like eigenfunctions $||y_i||$

$$(y_i, y_j) = \int_a^b y_i^*(x) y_j(x) p(x) dx = \|y_i\|^2 \delta_{ij}$$

- Orthonormal eigenfunctions

$$(f_i, f_j) = \int_a^b f_i^*(x) f_j(x) p(x) dx = \delta_{ij}$$

- Convert orthogonal eigenfunctions to orthonormal eigenfunctions

$$f_i = y_i / \|y_i\|$$

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Review Sturm-Liouville

General equation whose solutions provide orthogonal eigenfunctions

- Defined for a ≤ x ≤ b with p(x), q(x) and r(x) continuous and p(x) > 0
- (Homogenous) differential equation and boundary conditions shown below

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + [q(x) + \lambda p(x)]y = 0$$

$$k_1 y(a) + k_2 \left.\frac{dy}{dx}\right|_{x=a} = 0$$

$$\ell_1 y(b) + \ell_2 \left.\frac{dy}{dx}\right|_{x=b} = 0$$

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Review Sturm-Liouville Results

- Eigenvalues are real
- Eigenfunctions defined over a region a ≤ x ≤ b form an orthogonal set over that region.
- Eigenfunctions form a complete set over an infinite-dimensional vector space
- We can expand any function over the region in which the Sturm-Liouville problem is defined in terms of the eigenfunctions for that problem

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Review Eigenfunction Expansions

- Eigenfunction expansion formula $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$
- Equation for a_m coefficients in eigenfunction expansion of f(x)

$$a_m = \frac{(y_m, f)}{(y_m, y_m)} = \frac{\int_a^b p(x) y_m(x) f(x) dx}{\int_a^b p(x) y_m(x) y_m(x) dx}$$

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Review Expansion of f(x) = x

- Start with general equation for a_m
- Use $y_m = \sin(m\pi x)$ over $0 \le x \le 1$ which is a Sturm-Liouville solution
- Weight function p(x) = 1

$$a_m = \frac{(y_m, f)}{(y_m, y_m)} = \frac{\int_0^1 x \sin(m\pi x) dx}{\int_0^1 \sin^2(m\pi x) dx} = \frac{\int_0^1 x \sin(m\pi x) dx}{1/2} = \frac{2(-1)^{m+1}}{m\pi}$$

$$f(x) = x = \frac{2}{\pi}\left[\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \frac{\sin(4\pi x)}{4} + \cdots\right]$$

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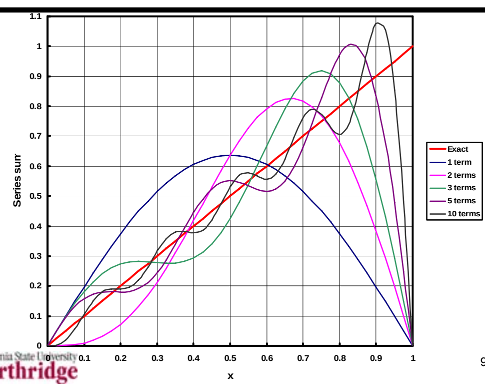
Review Partial Sums – Small



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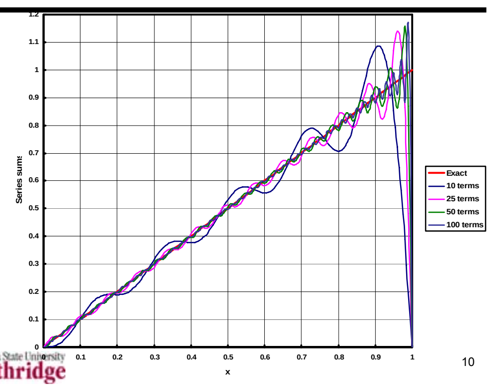
Review Partial Sums – Large



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Fourier Series

- Have same basic idea as eigenfunction expansions
  - Represent other functions, f(x), as a series of sines and cosines
  - Compute coefficients in a similar way to eigenfunction expansions
  - Fourier series based on periodicity of trigonometric functions

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Fourier Series

- Equations for series and coefficients defined for –L < x < L

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x) dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

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Basis for Fourier Series

- Applies to periodic functions
- Must be piecewise continuous
- Derivative must exist at all points in the period
- At discontinuity both left-hand and right-hand derivatives exist
- Fourier series converges to f(x)
- At discontinuity the series converges to average of f(x-) and f(x+)

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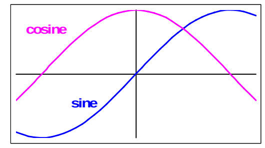
Even and Odd Functions

- Odd function: f(–x) = – f(x) (like sine)
- Even function g(–x) = g(x) (like cosine)



- For odd f(x) $\int_{-L}^{L} f(x)dx = 0$
- For even g(x) $\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$

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Even and Odd Functions II

- The product of an even function, g(x) = g(–x) and an odd function f(–x) = –f(x) is an odd function
- Proof: define h(x) = f(x) · g(x)
- h(–x) = f(–x) · g(–x) = –f(x) · g(x) = –h(x)
- Fourier coefficient $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$
- Since sine is an odd function, b_n = 0 if f(x) is even, i.e., f(–x) = f(–x)

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Even and Odd Functions III

- For odd functions, the Fourier series coefficients a_0 and a_n are zero $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
- Can have half-range expansions that consider only 0 < x < L
- Apply to non periodic functions to get "periodic extensions"

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Even and Odd Fourier Series

- We can use relations for even and odd functions to transform Fourier coefficient equations
- For odd functions, f(–x) = –f(x) we have only sine terms with coefficients b_n
- Even functions have only cosine (and constant) terms with coefficients a_0 and a_n
- Next chart shows series and coefficients

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Odd Function Sine Series

- f(x) is an odd function f(–x) = –f(x)
- Sine only series defined for –L ≤ x ≤ L
- b_n computed as integral from 0 to L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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Even Function Cosine Series

- f(x) is an even function f(–x) = f(x)
- Cosine only series defined for –L ≤ x ≤ L
- a_n evaluated by integral from 0 to L

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

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Half-interval series

- Based on equations for sine series for odd functions and cosine series for even functions
- Either the sine or cosine series can apply to any function
  - Can use sines to expand even functions
  - Can use cosines to expand odd functions
- Each series defined for 0 ≤ x ≤ L
- Behavior outside region 0 ≤ x ≤ L depends on the function

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Half-interval Series Example

- Last class computed series for f(x) = x in terms of sin(nπx/L) with L = 1 for 0 ≤ x ≤ L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

- Would get same coefficients from equations for Fourier sine series
- Get correct result for –L ≤ x ≤ L with periodic extensions

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2L(-1)^{n+1}}{n\pi}$$

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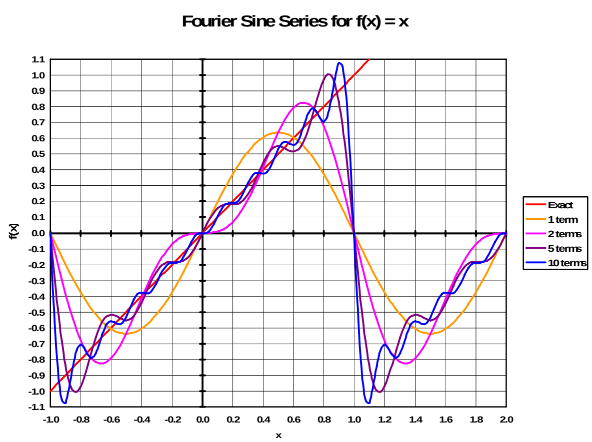

Cosine Series for f(x) = x

- Set f(x) = x in equations for a_0 and a_n

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} xdx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right) + \frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

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Cosine Series for f(x) = x

- Result for a_n is zero for even n values

$$a_n = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right) + \frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2[\cos(n\pi) - \cos(0)] + \left(\frac{L}{n\pi}\right)[L\sin(n\pi)$$

$$-(0)\sin(0)] = \frac{2L(\cos(n\pi)-1)}{(n\pi)^2} = -\frac{4L}{(n\pi)^2} \quad (odd\ n)$$

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Cosine Series for f(x) = x

- Final series, shown below, uses n = 2m+1 to get odd values of n only

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos\left(\frac{(2m+1)\pi x}{L}\right)$$

$$= \frac{L}{2} - \frac{4L}{\pi^2}\left[\cos\left(\frac{\pi x}{L}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{L}\right) + \cdots\right]$$

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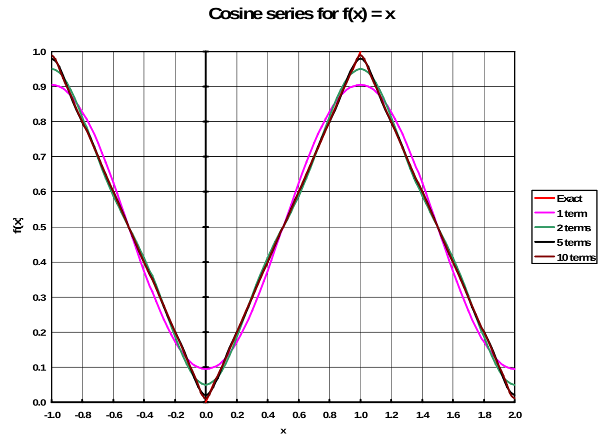

Complex Fourier Series

- Euler relationship $e^{ix} = \cos(x) + i \sin(x)$
- Setting x = -x gives $e^{-ix} = \cos(-x) + i \sin(-x) = \cos(x) - i \sin(x)$
- Get result for f(x) and coefficient c_n

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-inx}$$

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Summary

- Fourier series is alternative approach to Sturm-Liouville for developing series expansions in sines and cosines
- Based on periodic functions
- Can use half-interval expansions to show application to any function
- Basic approach similar to eigenfunction expansions: find the coefficients

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