Course Introduction and Eigenfunction Expansions

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Mechanical Engineering 501B

Seminar in Engineering Analysis

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California State University Northridge

Overview

- Course outline, schedule, grading, exams, homework, office hours, etc.
- Problems considered
- Eigenfunction expansions
  - Why do we care about this?
  - Sturm-Liouville problem
  - Complete orthogonal eigenfunctions
  - Examples of eigenfunction expansions
    - Sines and cosines
    - Bessel functions
    - Other eigenfunctions

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Course Structure

- ME 501AB is a one-year course in engineering and numerical analysis
  - Review 501A topics when required
- Look at solving problems once they are formulated
- Two overall goals
  - Understand advanced mathematical and computational approaches encountered in your work and future course work
  - Develop the ability to apply appropriate problem-solving skills

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Course Materials

- Web site: http://www.csun.edu/~lcaretto/me501b
- Lecture presentations supplemented (in some cases) by course notes
- Reading assignments in text should be done prior to class
- Download notes prior to class
- Homework assignments and solutions
- What's new section on home page for recent additions

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Instructor and Course Data

- lcaretto@csun.edu 818.677.6448
- Extensive email availability
- Office JD 3333, office hours MW 5 – 6 pm and TTh 2 – 3 pm other times by drop in, phone call, or appointment
- Texts: Kreyszig, *Advanced Engineering Mathematics*, and Hoffman, *Numerical Methods for Engineers and Scientists*
- Grading based on homework (10%), 2 midterms (50%) and a final (40%)
- See grading criteria in outline

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Course Objectives

- Understand and be able to apply analytical and numerical solutions of partial differential equations
- Read publications of applied engineering analysis and numerical analysis that involve ordinary and partial differential equations
  - Related topics include matrix operations and special functions such as Bessel functions

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Course objectives II

- Be familiar with algorithms and software packages for differential equations and understand their limitations
- Solve engineering problems using differential equations and understand when numerical solutions are required
- Understand differences between parabolic, hyperbolic and elliptic equations in both analytical and numerical approaches

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Course objectives III

- Understand methods used in numerical analysis and be able to apply them in simple cases
  - Finite difference approaches are simplest
  - Finite element approaches are used for complex geometries
  - Both convert differential equations to a system of algebraic equations that is solved using numerical approaches

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Galileo Galilei

(1564-1642)

You cannot teach people anything; you can only help them find it within themselves.

http://space.about.com/od/astronomyhistory/a/galileoquotes.htm

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Goals for this Course

- My goal is to help all students find within themselves sufficient knowledge of engineering analysis so that they will all get an A grade in the course
- What is your goal for this course?
- What will you do to achieve that goal?

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How to get your A

- Spend six to ten hours per week outside class studying for the course
- Prepare for lecture and be ready to ask questions
  - Read the assigned reading before class
  - Download, print, and review the lecture presentations before class
    - Use these as notes so that you can follow the lecture; write additional notes on these presentations

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How to Get your A, Part II

- Study with fellow students and try to answer each other's questions
- Do the homework assignments
- Contact me by email, telephone or office visits to ask questions
- Develop a good working relation with other members the class
  - Participate in class discussions

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What I will do to help

- Arrive at class a few minutes early to answer any questions you may have
- Give lectures that stress application of basics to problem solving
- Return homework and exams promptly so that you can learn from your errors
- Be available for questions in my office (visit or telephone) or email
  - Send entire class emails as appropriate

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Preliminary Assessment

- Designed to help instruction
- One set of questions on student background
- Second set of questions is ungraded quiz
- Take about 10 minutes for assessment
- Hand yours in when finished
  - Will call time when most students are done

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Kinds of Problems

- General case of a flux proportional to a gradient of some potential
  - Heat transfer: $q_x = -k\partial T/\partial x$
  - Mass diffusion: $j_x = -\rho D\ \partial\omega/\partial x$
  - Also applicable to simplified analyses of stress, velocity, and currents
- Conservation equation relates flow, which is flux times area to conserved quantity, Q: Net inflow = accumulation rate = $\Delta x\Delta y\Delta z\ \partial Q/\partial t$

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Energy Conservation Example

- Net heat input, $(q_x - q_{x+\Delta x})\Delta y\Delta z = \rho c_p\Delta x\Delta y\Delta z\ \partial T/\partial t$ (Here $dQ = \rho c_p dT$)
  - Divide by $\Delta x\Delta y\Delta z$ and take $\Delta x \to 0$ limit

$$-\frac{\partial q_x}{\partial x} = \lim_{\Delta x\to 0}\frac{q_x - q_{x+\Delta x}}{\Delta x} = \rho c_p\frac{\partial T}{\partial t}$$

  - Substitute $q_x = -k\partial T/\partial x$

$$\rho c_p\frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} = -\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x}$$

  - Multidimensional problems have terms in y and z directions like the x direction term

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Energy Example Continued

$$\rho c_p\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} = \nabla \bullet k\nabla T$$

- For constant k

$$\rho c_p\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = k\nabla^2 T$$

- "Del" operator in Cartesian coordinates

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\rho c_p\frac{\partial T}{\partial t} = k\nabla^2 T \qquad \alpha = \frac{k}{\rho c_p} \qquad \boxed{\frac{\partial T}{\partial t} = \alpha\nabla^2 T}$$

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Specific Examples

- One-, two- and three-dimensional transient, Cartesian problems

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \qquad \frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

- Steady problems (Laplace's equation)

$$\frac{d^2 T}{dx^2} = 0 \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

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Vector Spaces

- Abstract concept that generalizes usual vectors from mechanics
- Define linear independence
- Basis is a "complete set" of linearly independent "vectors" which can be used to express any vector in the space
- Vector dot product, **a**·**b**, generalized to inner product (**a**,**b**)
  - **a** and **b** orthogonal if (**a**,**b**) = 0
  - For functions, (f,g) = ∫fgdx

Orthogonal Functions

- Functions such as sin(nπx/L) form a vector space in the region 0 ≤ x ≤ L.
- The inner product, defined below, shows that this is a set of orthogonal functions

$$(f_n, f_m) = \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm} \qquad \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

- The set of functions at the right is orthonormal $g_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

The Sturm-Liouville Problem

- Why do we care about this problem?
  - Solution of partial differential equations (PDE) based on separation of variables
  - Obtain solutions to PDEs as product of ordinary differential equation (ODE) solutions
  - Want to obtain solution for general initial or boundary conditions, *e.g.* u(x,t=0) = f(x)
  - Will show that general solution to PDE has the form $u(x,t) = \Sigma_n C_n g_n(t) w_n(x)$ so that initial condition is $f(x) = \Sigma_n C_n g_n(0) w_n(x)$
  - Need to have functions $w_n(x)$ that allow us to express this initial condition easily

The Sturm-Liouville Problem II

- Solutions are easily used to obtain initial or boundary conditions in PDE solutions
- Solutions to SL provide complete set of orthogonal eigenfunctions than can be used to represent any other function

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + [q(x) + \lambda p(x)]y = 0$$

Homogenous ODE

  - Defined for a ≤ x ≤ b
  - p(x), q(x) and r(x) continuous
  - p(x) > 0

The Sturm-Liouville Problem III

- Boundary conditions at x = a and x = b
- At least one of k_1 or k_2 must be nonzero
- Similar requirement for ℓ_1 and ℓ_2

$$k_1 y(a) + k_2 \left.\frac{dy}{dx}\right|_{x=a} = 0$$

$$\ell_1 y(b) + \ell_2 \left.\frac{dy}{dx}\right|_{x=b} = 0$$

- Periodic boundary conditions y(a) = y(b) can replace those above
- If r(a) = 0 or r(b) = 0 only require y to be finite at boundary points where r = 0

Eigenvalue Problems

- Trivial solution to the Sturm-Liouville problem is y = 0
- Nontrivial solutions only hold for certain values of λ known as eigenvalues
- Matrix eigenvalues **Ax** = λ**x**
- Function eigenvalues defined in terms of a linear operator L: Lf = λf
  - λ is eigenvalue of operator L
  - f is eigenfunction of operator L
- General eigenvalue: Lf = λpf

Sturm-Liouville Eigenvalues

$$Ly = \left[\frac{d}{dx}\left(r(x)\frac{d}{dx}\right) + q(x)\right]y = -\lambda p(x)y$$

- Boundary condition is part of definition
- Note that y = 0 is solution
- Start example below to show boundary conditions restrict λ to certain values
  - p(x) = r(x) = 1 and q(x) = 0
  - Region defined as $0 \le x \le 1$ (a = 0, b = 1)
  - Differential equation: $d^2y/dx^2 + \lambda y = 0$
  - Boundary conditions y(0) = y(1) = 0

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Sturm-Liouville Example

- $d^2y/dx^2 + \lambda y = 0$ with y(0) = y(1) = 0
  - General solution is y = A $\sin(\lambda^{1/2}x)$ + B $\cos(\lambda^{1/2}x)$
  - For y(0) = 0 we must have B = 0
  - For y(1) = 0 we must have $\lambda^{1/2} = n\pi$
    - n is an integer ≥ 1
    - A = 0 or n = 0 give trivial solution that y = 0
- We call $\lambda^{1/2} = n\pi$ the eigenvalue and $\sin(\lambda^{1/2} x) = \sin(n\pi x)$ the eigenfunction
- Infinite set of eigens for n = 1, 2, …

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Self-Adjoint Operators

- The Sturm-Liouville operator belongs to a class of operators known as Hermitian or self-adjoint operators: (Lx,y) = (x,Ly)
- Define a Hermitian matrix: $\mathbf{H} = (\mathbf{H}^*)^T$
- For real-values matrices, $\mathbf{H} = \mathbf{H}^T$
- Column vector dot product is $\mathbf{x}^T\mathbf{y}$
  - Inner product notation: $(\mathbf{x},\mathbf{y}) = \mathbf{x}^T\mathbf{y}$
  - Can show that $(\mathbf{Hx},\mathbf{y}) = (\mathbf{x},\mathbf{Hy})$
- Self-adjoint is general kind of operator that includes matrix multiplication

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Self-adjoint Operator Results

- Eigenvalues are real
- Eigenfunctions form an orthogonal set over problem region $a \le x \le b$
- Eigenfunctions form a complete set if the vector space has a finite number of dimensions as in an n x n matrix
- Sturm-Liouville operator has a complete set of (orthogonal) eigenfunctions over an infinite-dimensional vector space
  - Can expand any function in terms of solutions to a Sturm-Liouville problem

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Orthogonal Eigenfunctions

- Defined in terms of inner product
- Norm of two like eigenfunctions $||y_i||$

$$(y_i, y_j) = \int_a^b y_i^*(x) y_j(x) p(x) dx = \|y_i\|^2 \delta_{ij}$$

- Orthonormal eigenfunctions have unit norm

$$(f_i, f_j) = \int_a^b f_i^*(x) f_j(x) p(x) dx = \delta_{ij}$$

- Convert orthogonal eigenfunctions to orthonormal eigenfunctions

$$f_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\left[\int_a^b y_i^*(x) y_i(x) p(x) dx\right]^{1/2}}$$

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Eigenfunction Expansions

- Sturm-Liouville solutions are complete sets of orthogonal eigenfunctions
- Can expand any function in terms of these eigenfunctions
- Valid region is $a \le x \le b$ defined for the original problem
- Problem is finding a_m in expansion for an arbitrary function, f(x)

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$$

Can start at m = 0 or m = 1

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Eigenfunction Expansions II

- Original expansion equation $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$
- Multiply by $p(x)y_n$ and take inner product

$$(y_n, f) = \int_a^b p(x) y_n(x) f(x) dx = \int_a^b p(x) y_n(x) \sum_{m=0}^{\infty} a_m y_m(x) dx$$

$$= \sum_{m=0}^{\infty} a_m \int_a^b p(x) y_n(x) y_m(x) dx = \sum_{m=0}^{\infty} a_m (y_n, y_m) = a_n (y_n, y_n)$$

- With orthogonal eigenfunctions, the infinite sum has only one nonzero term

$a_n = (y_n,f)/(y_n,y_n)$

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Remember These Formulas

- Finding the coefficients, a_m in the expansion $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$

$$a_n = \frac{(y_n, f)}{(y_n, y_n)} = \frac{\int_a^b p(x) y_n(x) f(x) dx}{\int_a^b p(x) y_n(x) y_n(x) dx}$$

- Note that letter used for subscript (n or m) is arbitrary as is starting index (0 or 1)

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Subscripts

- Different letters for subscripts can lead to confusion about subscripts on C
- Is it C_m or C_n or $C_{\text{something else}}$?
- What are differences in expressions in each column to the right?

$$f(x) = x^2 \qquad C_m = 1 - \cos(m\pi)$$
$$f(y) = y^2 \qquad C_n = 1 - \cos(n\pi)$$
$$f(a) = a^2 \qquad C_\alpha = 1 - \cos(\alpha\pi)$$

- There are none!

Each equation uses a dummy variable

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Example Expansion

- Expand f(x) = x using $y_m = \sin(m\pi x)$ over $0 \le x \le 1$
- Weight function p(x) = 1
- Can show $(y_m,y_m) = \int_0^1 \sin^2(m\pi x)dx = 1/2$
- Use equation for a_m from previous chart

$$a_m = \frac{(y_m, f)}{(y_m, y_m)} = \frac{\int_0^1 f(x)\sin(m\pi x) x dx}{\int_0^1 \sin^2(m\pi x) dx} = 2\int_0^1 f(x)\sin(m\pi x) x dx$$

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Expansion of f(x) = x Continued

$$a_m = 2\int_0^1 x \sin(m\pi x) dx = 2\left[\frac{1}{m^2\pi^2}\sin(m\pi x) - \frac{x}{m\pi}\cos(m\pi x)\right]_{x=0}^{x=1}$$

$$= \frac{2}{m^2\pi^2}[\sin(m\pi) - \sin(0)] - \frac{2}{m\pi}[\cos(m\pi) - 0] = \frac{-2\cos(m\pi)}{m\pi}$$

- $\cos(m\pi)$ = -1 or 1 if m is odd or even
- Usually written as $\cos(m\pi) = (-1)^m$
- Result is $a_m = -2(-1)^m/m\pi = 2(-1)^{m+1}/m\pi$
- What is our final expansion for f(x) = x?

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Expansion of f(x) = x Conclusion

$$f(x) = x = \sum_{m=1}^{\infty} a_m y_m(x) = \sum_{m=1}^{\infty} a_m \sin(m\pi x) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m\pi}\sin(m\pi x)$$

- What does series look like?
- Look at series sum for various terms

$$f(x) = x = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m\pi}\sin(m\pi x) = \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\sin(m\pi x)}{m}$$

$$f(x) = x = \frac{2}{\pi}\left[\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \frac{\sin(4\pi x)}{4} + \cdots\right]$$

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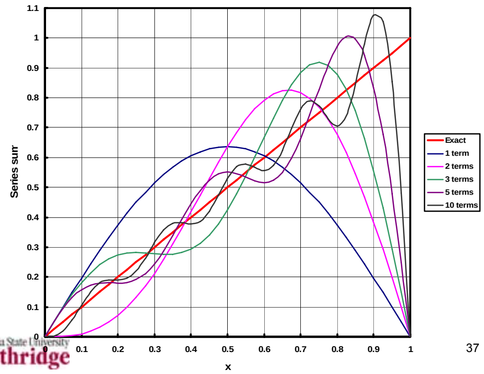
Partial Sums – Small Terms



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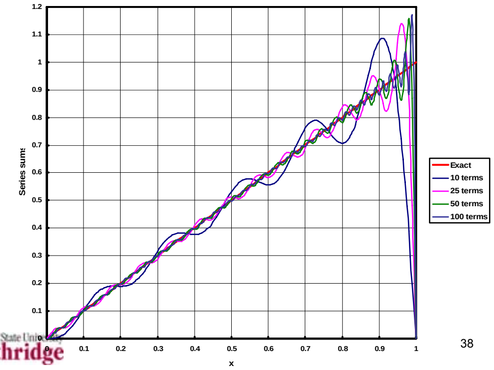
Partial Sums – Large Terms



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What kind of fit is this?

- Eigenfunction expansions have "mean convergence" $\|f_k - f\| \to 0$ as $k \to \infty$

$$f_k = \sum_{m=0}^{k} a_m y_m$$

$$\|f_k - f\| = \int_a^b \left(\sum_{m=0}^{k} a_m y_m - f(x) \right)^2 p(x)dx$$

- Bessel's inequality $$\sum_{m=0}^{k} a_m^2 \le \|f\|^2 = \int_a^b (f(x))^2 p(x)dx$$
- Parseval's equality $$\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b (f(x))^2 p(x)dx$$

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Sturm-Liouville Problems

- Geometries, coordinates, solutions
  - Rectangular, any, sines and cosines
  - Cylindrical, radial, Bessel functions
  - Spherical, radial, sin(λr)/r and cos(λr)/r
  - Spherical, azimuth angle, Legendre polynomials
  - Spherical, both angles, spherical harmonic functions
- Eigenfunction solutions differ with different boundary conditions

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Sines and Cosines

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \quad a \le x \le b \quad r = 1 \quad q = 0 \quad p = 1$$

$$y = A\sin\lambda x + B\cos\lambda x \quad For\ y(0) = y(L) = 0: y = A\sin\frac{n\pi x}{L}$$

$$For\ y(0) = y(L) = 0: \quad (y_n, y_m) = \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \frac{L}{2}\delta_{mn}$$

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \quad a \le x \le b \quad r = 1 \quad q = 0 \quad p = 1$$

$$y = A\sin\lambda x + B\cos\lambda x \quad For\ y'(0) = y(L) = 0: y = A\cos\frac{n\pi x}{L}$$

$$For\ y'(0) = y(L) = 0: \quad (y_n, y_m) = \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{L}{2}\delta_{mn}$$

= L if m = n = 0

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Eigenfunction Expansions

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) \qquad a_m = \frac{(y_m, f)}{(y_m, y_m)}$$

- Sine/cosine with y(0) = y(L) = 0 $$a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx$$
- Sine/cosine with y'(0) = y(L) = 0 $$a_m = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & m = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}dx & m \ne 0 \end{cases}$$

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Bessel Functions J_n and Y_n

$$\frac{d}{dx}x\frac{dy}{dx}+\left[-\frac{n^2}{x}+\lambda^2 x\right]y=0 \quad 0\le x\le R \quad r=x \quad q=-\frac{\nu^2}{x} \quad p=x$$

$$r(a)=r(0)=0 \quad No\,BC,\,but\; y\; is\; finite\; at\; x=0$$

$$y=AJ_n(\lambda x)+BY_n(\lambda x) \quad For\; y(R)=0: y_p=AJ_n\left(\frac{\alpha_{pn}x}{R}\right)$$

$$For\; y(R)=0: \quad (y_p,y_q)=\int_0^R xJ_n\left(\frac{\alpha_{pn}x}{R}\right)J_n\left(\frac{\alpha_{qn}x}{R}\right)=\|y_q\|^2\delta_{pq}$$

- n is an integer problem parameter
- $J_n(\alpha_{mn}) = 0$ for m = 1, 2, …, ∞ so α_{mn} are called the zeros of the Bessel function J_n

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More Eigenfunction Expansions

$$f(x)=\sum_{m=0}^{\infty}a_m y_m(x) \qquad a_m=\frac{(y_m,f)}{(y_m,y_m)}$$

- Bessel function with y(R) = 0 and arbitrary value n for parameter

$$a_q=\frac{\int_0^R xf(x)J_n\left(\frac{\alpha_{qn}x}{R}\right)dx}{\int_0^R x\left[J_n\left(\frac{\alpha_{qn}x}{R}\right)\right]^2 dx}$$

- Bessel function with y(R) = 0 and n = 0 for parameter

$$a_q=\frac{\int_0^R xf(x)J_0\left(\frac{\alpha_{q0}x}{R}\right)dx}{\frac{R^2}{2}J_1\left(\frac{\alpha_{q0}x}{R}\right)}$$

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Sphere Radial Coordinate x

$$\frac{d}{dx}x^2\frac{dy}{dx}+\lambda^2x^2y=0 \quad R_i\le x\le R_o \quad r=x^2 \quad q=0 \quad p=x^2$$

$$y=A\frac{\sin\lambda x}{x}+B\frac{\cos\lambda x}{x} \quad Consider\; BC\; y(R_i)=y(R_o)=0$$

$$y_n=\frac{\sin\lambda_n(x-R_i)}{x} \qquad \lambda_n=\frac{n\pi}{R_o-R_i}$$

$$(y_n,y_m)=\int_{R_i}^{R_o}x^2\left[\frac{1}{x}\sin\left(\frac{n\pi(x-R_i)}{R_o-R_i}\right)\right]\left[\frac{1}{x}\sin\left(\frac{m\pi(x-R_i)}{R_o-R_i}\right)\right]dx$$

$$(y_n,y_m)=\delta_{mn}\|y_n\|^2=\delta_{mn}\int_{R_i}^{R_o}x^2\left[\frac{1}{x}\sin\left(\frac{n\pi(x-R_i)}{R_o-R_i}\right)\right]^2dx=\delta_{mn}\frac{R_o-R_i}{2}$$

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More Eigenfunction Expansions

$$f(x)=\sum_{m=0}^{\infty}a_m y_m(x) \qquad a_m=\frac{(y_m,f)}{(y_m,y_m)}$$

- Spherical shell $y(R_i) = y(R_o) = 0$

$$a_n=\frac{\int_{R_i}^{R_o}x^2f(x)\frac{1}{x}\sin\left(\frac{n\pi(x-R_i)}{(R_o-R_i)}\right)dx}{\int_{R_i}^{R_o}x^2\left[\frac{1}{x}\sin\left(\frac{n\pi(x-R_i)}{(R_o-R_i)}\right)\right]^2dx}$$

$$=\frac{2}{R_o-R_i}\int_{R_i}^{R_o}x^2f(x)\frac{1}{x}\sin\left(\frac{n\pi(x-R_i)}{(R_o-R_i)}\right)dx$$

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Sphere Azimuth Angle, ϕ

- Based on transformation that $x = \cos\phi$
  - $0 \le \phi \le \pi$ becomes $1 \ge x \ge -1$

$$\frac{d}{dx}\left(1-x^2\right)\frac{dy}{dx}+n(n+1)y=0 \quad -1\le x\le 1 \quad r=1-x^2 \quad q=0 \quad p=1$$

$$\lambda=n(n+1) \quad r(-1)=r(1)=0 \quad only\; require\; y\; finite\; at\; x=\pm 1$$

$$y_n=P_n(x)=\sum_{m=0}^{\mathrm{int}(n/2)}\frac{(-1)^m(2n-2m)!x^{n-2m}}{2^n n!(n-m)!(n-2m)!} \quad (Legendre\; polynomials)$$

$$(y_n,y_m)=\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2\delta_{mn}}{2n+1}$$

$$a_n=\frac{(y_n,f)}{(y_n,y_n)}=\frac{\int_{-1}^{1}f(x)P_n(x)dx}{\int_{-1}^{1}[P_n(x)]^2dx}=\int_{-1}^{1}f(x)P_n(x)dx=\frac{2\delta_{mn}}{2n+1}\int_{-1}^{1}f(x)P_n(x)dx$$

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Summary

- Solution to Sturm-Liouville problem is set of orthogonal eigenfunctions, $y_m(x)$, that can be used to express any function, f(x)
  - Homogenous equation/boundary conditions
  - Problem/expansion region is $a \le x \le b$
  - Weighting function p(x) from SL equation

$$f(x)=\sum_{m=0}^{\infty}a_m y_m(x) \qquad a_m=\frac{(y_m,f)}{(y_m,y_m)}=\frac{\int_a^b p(x)y_m(x)f(x)dx}{\int_a^b p(x)y_m(x)y_m(x)dx}$$

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