California State University Northridge

College of Engineering and Computer Science
Mechanical Engineering Department

***Mechanical Engineering 501A***
***Seminar in Engineering Analysis***

Fall 2017 Number: 15966 Instructor: Larry Caretto

# Solutions to Second Midterm Exam (November 15, 2017)

**1. (50 points) Find the solution to the differential equation below without using Laplace transforms. Use the boundary conditions that y(0) = 1, y'(0) = 0, and y(1) = 0.**

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 2e^x - 2x$$

We first find the solution to the homogenous equation: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

The indicial equation, $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$, has three identical roots: -1, -1, -1. Thus, the homogenous solution is solution is $y_H = \left(A + Bx + Cx^2\right) e^{-x}$. The particular solution has the form $y_P = De^x + Ex + F$. Differentiating the particular solution three times and substituting the results into the differential equation gives the following equation.

$De^x + 3De^x + 3De^x + 3E + De^x + Ex + F = 8e^x - 2x - 1$

Equating coefficients of like terms gives the following equations for D, E, and F:

Exponential terms: D + 3D + 3D + D = 8D = 8. So D = 1.

Linear x terms: E = -2

Constant terms: 3E + F = -1. So F = -1 – 3E = -1 – 3(-2) = 5.

This gives the total solution, the sum of the homogenous and particular solution as follows.

$$y = y_H + y_P = \left(A + Bx + Cx^2\right)e^{-x} + e^x - 2x + 5$$

The boundary conditions are used to find the constants A, B, and C.

y(0) = 1 gives $y(0) = \left[A + B(0) + C(0)^2\right]e^{-0} + e^0 - 2(0) + 5 = A + 1 + 5 = 1 \quad \Rightarrow \quad A = -5$

y(1) = 0 gives $y(1) = \left[A + B(1) + C(1)^2\right]e^{-1} + e^1 - 2(1) + 5 = 0 \qquad (A + B + C) + e^2 + 3e^1 = 0$

The first derivative of the solution is $y' = (B + 2Cx)e^{-x} - \left(A + Bx + Cx^2\right)e^{-x} + e^x - 2$; so the boundary condition that y'(0) = 0 gives the following equation:

$$y'(0) = \left[B + 2C(0) - A - B(0) - C(0)^2\right]e^{-0} + e^0 - 2 = B - A + 1 - 2 = 0 \quad \Rightarrow \quad -A + B = 1$$

We already know that A = -5, so we have B = A + 1 = -5 + 1 = -4

Substituting A = -5 and B = -4 into the y(1) = 0 boundary condition gives the following result.

$$(A + B + C) + e^2 + 3e = (-5 - 4 + C) + e^2 + 3e = 0 \Rightarrow C = 9 - 3e - e^2 = -6.5439016...$$

So, the solution to our differential equation and boundary conditions is

$$\boxed{y = \left[-5 - 4x + \left(9 - 3e - e^2\right)x^2\right]e^{-x} + e^x - 2x + 5}$$

**2.(45 points)** Find the solution to the differential equation using Laplace transforms. Use the boundary conditions that y(0) = 1, y'(0) = -1.

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=10\sin t$$

Using the transform tables from Kreyszig we find the following equation for Y(s), where we will use the general frequency, ω, for sin(ωt) although ω is 1 in this problem.

$$s^2Y(s)-sy(0)-y'(0)+3[sY(s)-y(0)]+2Y(s)=10\frac{\omega}{s^2+\omega^2}$$

Substituting the initial conditions, factoring terms, and solving for Y(s) gives

$$s^2Y(s)-s(1)-(-1)+3[sY(s)-1]+2Y(s)=[s^2+3s+2]Y(s)=s-1+3+\frac{10\omega}{s^2+\omega^2}$$

$$[s^2+3s+2]Y(s)=(s+2)(s+1)Y(s)=s+2+\frac{10\omega}{s^2+\omega^2}$$

$$Y(s)=\frac{s+2}{(s+2)(s+1)}+\frac{10\omega}{(s+2)(s+1)(s^2+\omega^2)}=\frac{1}{s+1}+\frac{10\omega}{(s+2)(s+1)(s^2+\omega^2)}$$

We have to use partial fractions on the final term. This is done as follows using the special rule for complex factors.

$$\frac{10\omega}{(s+2)(s+1)(s^2+\omega^2)}=\frac{A}{(s+2)}+\frac{B}{(s+1)}+\frac{Cs+D}{(s^2+\omega^2)}$$

Before solving for A, B, C, and D, we can substitute this partial fraction result into our equation for Y(s) to get the following equation.

$$Y(s)=\frac{1}{s+1}+\frac{A}{(s+2)}+\frac{B}{(s+1)}+\frac{Cs+D}{(s^2+\omega^2)}$$

We can get the find the various terms in this Y(s) equation in the transform table and write our solution for y(t) as follows.

$$y(t)=e^{-t}+Ae^{-2t}+Be^{-t}+Ccos(\omega t)+\frac{D}{\omega}sin(\omega t)$$

We now apply ourselves to the task of using the method of partial fractions to find the constants A, B, C, and D from the following equation, copied from above.

$$\frac{10\omega}{(s+2)(s+1)(s^2+\omega^2)}=\frac{A}{(s+2)}+\frac{B}{(s+1)}+\frac{Cs+D}{(s^2+\omega^2)}$$

Multiplying this equation by the denominator of the fraction on the left and factoring the result to get the factors for like powers of s gives:

$$10\omega=A(s+1)(s^2+\omega^2)+B(s+2)(s^2+\omega^2)+(Cs+D)(s+2)(s+1)$$

$$10\omega=A(s^3+s^2+s\omega^2+\omega^2)+B(s^3+2s^2+s\omega^2+2\omega^2)+(Cs+D)(s^2+2s+2)$$

$$10\omega=s^3(A+B)+s^2(A+2B)+s\omega^2(A+B)+(A+2B)\omega^2+(Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D)$$

$$10\omega=s^3(A+B+C)+s^2(A+2B+2C+D)+s[\omega^2(A+B)+2C+2D]+(A+2B)\omega^2+2D$$

Equating like powers of s on both sides of the equation gives the following set of equations to be solved for A, B, C, and D.

$s^3$ terms: A + B + C = 0

$s^2$ terms: A + 2B + 2C + D = 0

**s terms:** $\omega^2(A+B)+2C+2D=0$

$s^0$ terms: $(A+2B)\omega^2 + 2D = 10\omega$

Subtracting the $s^3$ equation from all other equations after multiplying it by a factor of 1 (for the second equation) and by $\omega^2$ for the other equations gives our equation set as follows:

$s^3$ terms: A + B + C = 0

$s^2$ terms: B + C + D = 0

s terms: $(2-\omega^2)C + 2D = 0$

$s^0$ terms: $B\omega^2 - C\omega^2 + 2D = 10\omega$

Multiplying the modified $s^2$ equation by $\omega^2$ and subtracting from the $s^0$ equation gives a revised $s^0$ equation in the following equation set.

$s^3$ terms: A + B + C = 0

$s^2$ terms: B + C + D = 0

**s terms:** $(2-\omega^2)C + 2D = 0$

$s^0$ terms: $-2C\omega^2 + 2(1-\omega^2)D = 10\,\omega$

Multiplying the s equation by $(1-\omega^2)$and subtracting it from the $s^0$ equation gives the following set of equations.

**$s^3$ terms:** A + B + C = 0

**$s^2$ terms:** B + C + D = 0

**s terms:** $(2-\omega^2)C + 2D = 0$

**$s^0$ terms:** $-[2\omega^2 + (1-\omega^2)(2-\omega^2)]C = 10\omega$

We can now back substitute in the above set of equations to find the values of A, B, C, and D. From the $s^0$ equation we find

$$C = \frac{-10\omega}{2\omega^2 + (1-\omega^2)(2-\omega^2)}$$

Substituting this into the equation for the s terms we can find D.

$$D = \frac{-(2-\omega^2)C}{2} = \frac{10\omega(2-\omega^2)}{4\omega^2 + 2(1-\omega^2)(2-\omega^2)} = \frac{5\omega(2-\omega^2)}{2\omega^2 + (1-\omega^2)(2-\omega^2)}$$

We next find B for the equation for the $s^2$ terms.

$$B = -C - D = -\frac{-10\omega}{2\omega^2 + (1-\omega^2)(2-\omega^2)} - \frac{5\omega(2-\omega^2)}{2\omega^2 + (1-\omega^2)(2-\omega^2)} = \frac{5\omega^3}{2\omega^2 + (1-\omega^2)(2-\omega^2)}$$

Finally, we can obtain A from the equation for the $s^3$ terms.

$$A = -B - C = -\frac{5\omega^3}{2\omega^2 + (1-\omega^2)(2-\omega^2)} - \frac{-10\omega}{2\omega^2 + (1-\omega^2)(2-\omega^2)} = \frac{-5\omega^3 + 10\omega}{2\omega^2 + (1-\omega^2)(2-\omega^2)}$$

We can now use these values of A, B, C, and D in the equation that we previously wrote for our solution.

$$y(t) = e^{-t} + Ae^{-2t} + Be^{-t} + Ccos(ct) + \frac{D}{\omega}sin(\omega t)$$
$$= e^{-t} + \frac{10\omega - 5\omega^3}{2\omega^2 + (1-\omega^2)(2-\omega^2)}e^{-2t} + \frac{5\omega^3}{2\omega^2 + (1-\omega^2)(2-\omega^2)}e^{-t}$$
$$- \frac{10\omega}{2\omega^2 + (1-\omega^2)(2-\omega^2)}cos(\omega t) + \frac{5\omega(2-\omega^2)}{2\omega^2 + (1-\omega^2)(2-\omega^2)}\frac{1}{\omega}sin(\omega t)$$

In the problem given $\omega = 1$, so the solution is

$$y(t) = e^{-t} + \frac{5}{2}e^{-2t} + \frac{5}{2}e^{-t} - 5cos(t) + \frac{5}{2}sin(t)$$

**3.(10 points) Find the solution to the differential equations below. Use the boundary conditions that y(1) = 0 and y (2) = 1 for each equation. Use the table below to obtain your boundary conditions.**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 4\right)y = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 0.25\right)y = 0$$

| $\nu$ | -4 | -2 | -0.5 | -0.25 | 0.25 | 0.5 | 2 | 4 |
|---|---|---|---|---|---|---|---|---|
| $J_\nu(x=1)$ | 0.003 | 0.115 | 0.431 | 0.669 | 0.752 | 0.671 | 0.115 | 0.003 |
| $J_\nu(x=2)$ | 0.034 | 0.353 | -0.235 | 0.004 | 0.398 | 0.513 | 0.353 | 0.034 |
| $Y_\nu(x=1)$ | -33.278 | -1.651 | 0.671 | 0.394 | -0.194 | -0.431 | -1.651 | -33.278 |
| $Y_\nu(x=2)$ | -2.766 | -0.617 | 0.513 | 0.559 | 0.393 | 0.235 | -0.617 | -2.766 |

The first equation is Bessel's equation with $\nu$ = n = 2, an integer. So, the solutions are y = $AJ_2(x)$ + $BY_2(x)$. To match the first boundary condition we must have y(1) = $AJ_2(1)$ + $BY_2(1)$ = 0.115A + B(-1.651) = 0 so B = 0.115A/1.651 = 0.06965A. The second boundary condition, y(2) = 1 = $AJ_2(2)$ + $BY_2(2)$ = 0.353A + B(-0.617) = 0.353A + 0. 0.06965A(-0.617) = 0.3100A. So A = 1/.3100 = 3.226 and B = 0.06965A = 0.06965(3.226) = 0.2247. So, the solution to the first equation is **y = 3.226$J_2$(x) + 0.2247$Y_2$(x)**.

The second equation is Bessel's equation with $\nu$ = 0.5, a non-integer. So, the solutions are y = $AJ_{0.5}(x)$ + $BJ_{-0.5}(x)$. To match the first boundary condition we must have y(1) = $AJ_{0.5}(1)$ + $BJ_{-0.5}(1)$ = 0 = 0.671A + 0.431B = 0 so B = -0.671A/0.431 = -1.557A. The second boundary condition, y(2) =1 = $AJ_{0.5}(2)$ + $BJ_{-0.5}(2)$ = 0.513A + B(-0.235) = 0.513A + -1.557A(-0.235) = 0.87886A = 1. So A = 1/0.87886 = 1.138 and B = -1.557A = -1.557(1.138) = -1.771. The solution to the second equation is **y = 1.138$J_{0.5}$(x) – 1.771$J_{-0.5}$(x)**.