College of Engineering and Computer Science
Mechanical Engineering Department

Mechanical Engineering 501A
Seminar in Engineering Analysis

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November 29 Homework Solutions

1. **Solve the problem y' = -0.2xy with y(0) = 1 for ten steps, with h = 0.1, using the Adams-Moulton method. Compare the errors with the exact solution. Use the Runge-Kutta method to get the starting values.**

The exact solution can be found by separating the differential equation. This gives dy/y = -0.2xdx so that ln y = -0.1 x^2 + C. Setting y(0) = 1 gives ln(1) = 0 + C or C = 0. Thus, the exact solution is y = $e^{-0.1x2}$. The Adam-Moulton method uses the following algorithm. First, we predict a new value at the forward (i+1) step.

$$y_{i+1}^{P} = y_i + \frac{h}{24}\left[-9f_{i-3} + 37f_{i-2} - 59f_{i-1} + 55f_i\right]$$

The derivative at x_{i+1}, computed with this predicted y value is then used to obtain the final (corrected) value of y_{i+1} by the following equation.

$$y_{i+1}^{C} = y_i + \frac{h}{24}\left[f_{i-2} - 5f_{i-1} + 19f_i + 9f(x_{i+1}, y_{i+1}^{P})\right]$$

Since this algorithm depends on having values at three previous steps, we need a starting method, for which we can use the fourth-order Runge-Kutta algorithm, shown below.

$$y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \qquad x_{i+1} = x_i + h_{i+1}$$

$$k_1 = h_{i+1} f(x_i, y_i)$$

$$k_2 = h_{i+1} f\left(x_i + {h_{i+1}}/{2}, y_i + {k_1}/{2}\right)$$

$$k_3 = h_{i+1} f\left(x_i + {h_{i+1}}/{2}, y_i + {k_2}/{2}\right)$$

$$k_4 = h_{i+1} f(x_i + h_{i+1}, y_i + k_3)$$

For this problem, $f_i = -0.2x_iy_i$, and we use this in computing the various f_i terms in both algorithms. Here we use the Runge-Kutta algorithm to get enough values of f_i to start the Adams-Moulton method. The details of this calculation are shown in the table below. In this table, the initial conditions are shown in the first row. This row shows the k values required to compute y_1, the first value computed in the Runge-Kutta solution.

The computed and actual values of y_1 are shown in the second row. This process continues until the final row in the table contains the value of y_3. Which gives sufficient x-y values to begin the Adams procedure.

i	x_i	y_i	exact y_i	error	f_i	k_1	k_2	k_3	k_4
0	0	1	1	0	0	0	-0.004	-0.00399	-0.00797
1	0.2	0.996008	0.996008	1.07E-11	-0.03984	-0.00797	-0.0119	-0.01188	-0.01575
2	0.4	0.984127	0.984127	1.04E-10	-0.07873	-0.01575	-0.01953	-0.01949	-0.02315
3	0.6	0.96464	0.96464	3.38E-10	-0.11576				

The values of y_3 and x_3 in this table are used to compute f_3, which is required in the Adams-Moulton procedure. The first step in this procedure is shown in detail below. Here, we take the general formulas on the previous page and apply them for i = 3.

$$y_4^P = y_3 + \frac{h}{24}\left[-9f_0 + 37f_1 - 59f_2 + 55f_3\right] = 0.960795 + {}^{0.2}\!/_{24}\left[-9(0)\right.$$
$$\left.+37(-0.03984) - 59(-0.07873) + 55(-0.11576)\right] = 0.93801$$

We can use this predicted value to compute the estimated derivate at point i + 1, using the formula that f = -0.2xy for the differential equation considered here.

$$f\left(x_{i+1}, y_{i+1}^P\right) = -0.2x_{i+1}y_{i+1}^P = (-0.2)(0.8)(0.93801) = -0.15008$$

With this derivative value, we can apply the corrector equation to compute the final value for y_4. (Here again we apply the general formula from the previous page for i = 3.)

$$y_4^C = y_3 + \frac{h}{24}\left[f_1 - 5f_2 + 19f_3 + 9f(x_4, y_4^P)\right] = 0.960795 + {}^{0.2}\!/_{24}\left[(-0.03984)\right.$$
$$\left.-5(-0.07873) + 19(-0.11576) + 9(-0.15008)\right] = 0.936833$$

The results of the previous Runge-Kutta calculations, the calculation to get y_4 and the results of all subsequent Adams steps are shown in the table below. Note that each new step in the Adams method starts with an evaluation of the derivative, f_n, based on the values of x_n and y_n^C. The derivative evaluation based on the predictor is not used to start the new step.

i	x_i	y_i	exact y_i	error	f_i	y_{i+1}^P	$f\left(x_{i+1}, y_{i+1}^P\right)$
0	0	1	1	0	0		
1	0.2	0.996008	0.996008	1.07E-11	-0.03984		
2	0.4	0.984127	0.984127	1.04E-10	-0.07873		
3	0.6	0.96464	0.964640	3.38E-10	-0.11576	0.93801	-0.15008
4	0.8	0.938004	0.938005	5.7E-07	-0.15008	0.904844	-0.18097
5	1	0.904836	0.904837	1.31E-06	-0.18097	0.865896	-0.20781
6	1.2	0.865886	0.865888	2.18E-06	-0.20781	0.822021	-0.23017
7	1.4	0.822009	0.822012	3.13E-06	-0.23016	0.77415	-0.24773
8	1.6	0.774138	0.774142	4.07E-06	-0.24772	0.723258	-0.26037
9	1.8	0.723245	0.72325	4.96E-06	-0.26037	0.670327	-0.26813
10	2	0.670314	0.67032	5.73E-06	-0.26813	0.616318	-0.27118
11	2.2	0.616307	0.616313	6.32E-06	-0.27118	0.562146	-0.26983
12	2.4	0.562136	0.562142	6.69E-06	-0.26983	0.508649	-0.2645
13	2.6	0.508641	0.508648	6.82E-06	-0.26449		

2. Solve the problem y' = ty^2 with y(0) = 1 from t = 0 to t = 1 using h = 0.2 and h = 0.1. Use the exact solution y = -2/(t^2 – 2/y$_0$) to compute the ratio of the errors for the two step sizes at t = 1. Use the implicit Euler method with time linearization which is given by the following equation $y_{n+1} - y_n = \left[hf_n + h^2 \left.\frac{\partial f}{\partial t}\right|_n \right] \Big/ \left[1 - h \left.\frac{\partial f}{\partial y}\right|_n \right]$

For this problem, f = ty^2, ∂f/∂t = y^2 and ∂f/∂y = 2ty. Substituting these values into the general implicit Euler expression with time linearization gives the following equation to be solved for this problem.

$$y_{n+1} = y_n + \frac{ht_n y_n^2 + h^2 y_n^2}{1 - h2t_n y_n}$$

The first two steps with h = 0.2 give the following results.

$$y_1 = y_0 + \frac{ht_0 y_0^2 + h^2 y_0^2}{1 - 2ht_0 y_0} = 1 + \frac{(0.2)(0)(1)^2 + (0.2)^2(1)^2}{1 - 2(0.2)(0)(1)} = 1.04$$

$$y_2 = y_1 + \frac{ht_1 y_1^2 + h^2 y_1^2}{1 - 2ht_1 y_1} = 1 + \frac{(0.2)(0.2)(1.04)^2 + (0.2)^2(1.04)^2}{1 - 2(0.2)(0.2)(1.04)} = 1.13438$$

The table below shows the results from the two steps above as well as the remaining steps to the end of the integration at t = 1. At this point the exact solution is y = -2/(1 – 2) = 2 and the error is 1.0829

n	t	y	f	df/dt	df/dy
0	0	1	0	1	0
1	0.2	1.04	0.21632	1.0816	0.416
2	0.4	1.13438	0.514728	1.286819	0.907504
3	0.6	1.323041	1.050262	1.750437	1.587649
4	0.8	1.733417	2.403789	3.004736	2.773468
5	1	3.082931	9.504466	9.504466	6.165863

Repeating the integration with h = 0.1 gives the following result.

n	t	y	f	df/dt	df/dy
0	0	1	0	1	0
1	0.1	1.01	0.10201	1.0201	0.202
2	0.2	1.030823	0.212519	1.062595	0.412329
3	0.3	1.064071	0.339674	1.132248	0.638443
4	0.4	1.11245	0.495018	1.237545	0.88996
5	0.5	1.180372	0.696639	1.393278	1.180372
6	0.6	1.275157	0.975615	1.626025	1.530188
7	0.7	1.409542	1.390766	1.986809	1.973359
8	0.8	1.607564	2.067409	2.584261	2.572102
9	0.9	1.920685	3.320128	3.689031	3.457233
10	1	2.484519	6.172833	6.172833	4.969037

In this case the error at t = 1 is 2.484519 – 2 = 0484519 so that the ratio of the errors for the two step sizes is 0484519/1.0829 = .44 which is about the value of 0.5 that we expect for this first-order method when we cut the step size in half.

3. Solve the problem $y' = ty^2$ with $y(0) = 1$ from t = 0 to t = 1 using h = 0.2 and h = 0.1. Use the exact solution $y = -2/(t^2 – 2/y_0)$ to compute the ratio of the errors for the two step sizes at t = 1. Use the implicit Euler method with Newton iteration in which the implicit Euler method is written as one equation in one unknown, $g(y_{n+1}) = 0$. One then applies the usual formula for Newton iteration, $y_{n+1}^{(m+1)} = y_{n+1}^{(m)} - g\left(y_{n+1}^{(m)}\right) \big/ g'\left(y_{n+1}^{(m)}\right)$, where g' denotes dg/dy_{n+1}. This iteration is used at each new time step, n + 1, to advance from iteration m to iteration m+1, until a satisfactory convergence criterion is reached. Use a convergence criterion that $\left|y_{n+1}^{(m+1)} - y_{n+1}^{(m)}\right| \le 10^{-5}\left|y_{n+1}^{(m+1)}\right|$ for this problem.

For the given problem, $y' = ty^2$, the implicit Euler equation becomes $y_{n+1} = y_n + ht_{n+1}y^2_{n+1}$. Writing this in the form $g(y_{n+1}) = 0$ and taking dg/dy_{n+1} gives.

$$g(y_{n+1}\ ) = y_{n+1} – y_n – ht_{n+1}y^2_{n+1} = 0 \qquad dg/dy_{n+1}\ ) = 1 – 2ht_{n+1}y_{n+1}$$

This gives the following iteration process for each step.

$$y_{n+1}^{(m+1)} = y_{n+1}^{(m)} - \frac{y_{n+1}^{(m)} - y_n - ht_{n+1}\left(y_{n+1}^{(m)}\right)^2}{1 - 2ht_{n+1}y_{n+1}^{(m)}}$$

For the first step we will use an initial guess that $y_1^{(0)} = y_0 = 1$. This gives the following iteration steps for y_1.

$$y_1^{(1)} = y_1^{(0)} - \frac{y_1^{(0)} - y_0 - ht_1\left(y_1^{(0)}\right)^2}{1 - 2ht_1y_1^{(0)}} = 1 - \frac{1 - 1 - (0.2)(0.2)(1)^2}{1 - 2(0.2)(0.2)(1)} = 1.043478$$

$$y_1^{(2)} = y_1^{(1)} - \frac{y_1^{(1)} - y_0 - ht_1\left(y_1^{(1)}\right)^2}{1 - 2ht_1y_1^{(1)}} = 1.043478 - \frac{1 - 1 - (0.2)(0.2)(1.043478)^2}{1 - 2(0.2)(0.2)(1.043478)} = 1.043560762$$

$$y_1^{(2)} = y_1^{(1)} - \frac{y_1^{(1)} - y_0 - ht_1\left(y_1^{(1)}\right)^2}{1 - 2ht_1y_1^{(1)}} = 1.043560762 - \frac{1 - 1 - (0.2)(0.2)(1.043560762)^2}{1 - 2(0.2)(0.2)(1.043560762)} = 1.043560763$$

At this point our iterations have a relative error of less than 3×10^{-10} and can be considered converged. Thus, our value of y_1 is 1.04356. For subsequent iterations we can get the initial guess by assuming that that the value of Δy for the new time step is the same as the value for the time step just completed. This gives

$$y_{n+1}^{(0)} - y_n = y_n - y_{n-1} \quad \Rightarrow \quad y_{n+1}^{(0)} = 2y_n - y_{n-1}$$

Using this process, our initial guess for the next time step is found as follows.

$$y_2^{(0)} = 2(1.04356) - 1 = 1.08712$$

With this initial guess, the iterations for y_2 proceed as follows.

$$y_2^{(1)} = y_2^{(0)} - \frac{y_2^{(0)} - y_2 - ht_2\left(y_2^{(0)}\right)^2}{1 - 2ht_2 y_2^{(0)}} = 1.08712 - \frac{1 - 1 - (0.2)(0.4)(1.08712)^2}{1 - 2(0.2)(0.4)(1.08712)} = 1.14884$$

$$y_2^{(2)} = y_2^{(1)} - \frac{y_2^{(1)} - y_2 - ht_2\left(y_2^{(1)}\right)^2}{1 - 2ht_2 y_2^{(1)}} = 1.14884 - \frac{1 - 1 - (0.2)(0.4)(1.14884)^2}{1 - 2(0.2)(0.4)(1.14884)} = 1.14921667$$

$$y_2^{(3)} = y_2^{(2)} - \frac{y_2^{(2)} - y_2 - ht_2\left(y_2^{(2)}\right)^2}{1 - 2ht_2 y_2^{(2)}} = 1.14921667 - \frac{1 - 1 - (0.2)(0.4)(1.14921667)^2}{1 - 2(0.2)(0.4)(1.14921667)} = 1.14921668$$

We have another converged solution so that we now have $y_2 = 1.149217$. The remaining steps of iterations and values for y are shown in the table below. In this table, y_n represents the value in the final iteration that we use for the result of our numerical integration and $y_n^{(m)}$ represents the value of y at iteration m for integration step n. At the final step, where $t_n = 5$, the iterations do not converge. For this problem, the nonlinear equation for $g(y_{n+1})$ is a quadratic equation that can be solved exactly. Using the quadratic solution and plugging in values for the step from t = 0.8 to t = 1 gives a complex number as shown below. Thus, the explicit Euler method does not give a valid result for this step size at t = 1.

$$y_{n+1} = \frac{-1 \pm \sqrt{1^2 - 4\left(-t_{n+1}h\right)\left(-y_n\right)}}{2\left(-t_{n+1}h\right)} \quad \Rightarrow \quad y_5 = \frac{-1 \pm \sqrt{1^2 - 4\left(-t_5 h\right)\left(-y_4\right)}}{2\left(-t_5 h\right)}$$

$$y_5 = \frac{-1 \pm \sqrt{1^2 - 4[-(1)(0.2)(2.047183)]}}{2[-(1)(0.2))]} = \frac{-1 \pm \sqrt{-0.6377468}}{-0.4} = 2.5 \pm 1.99647625i$$

n	t_n	y_n	m	$y_n^{(m)}$	$g(y_n^{(m)})$	$g'(y_n^{(m)})$
0	0	1				
1	0.2		0	1	-0.04	0.92
1	0.2		1	1.043478261	-7.56144E-05	0.916522
1	0.2		2	1.043560762	-2.7226E-10	0.916515
1	0.2	1.043561	3	1.043560763	0	0.916515
2	0.4		0	1.08712153	-0.050985894	0.826061
2	0.4		1	1.14884326	-0.000304766	0.816185
2	0.4		2	1.14921667	-1.11544E-08	0.816125
2	0.4	1.149217	3	1.14921668	0	0.816125
3	0.6		0	1.2548726	-0.08330871	0.698831
3	0.6		1	1.3740842	-0.001705369	0.67022
3	0.6		2	1.37662869	-7.76933E-07	0.669609
3	0.6	1.37663	3	1.37662985	-1.61593E-13	0.669609
4	0.8		0	1.60404302	-0.184259472	0.486706
4	0.8		1	1.98262759	-0.022932205	0.365559
4	0.8		2	2.04535945	-0.000629646	0.345485
4	0.8		3	2.04718195	-5.3144E-07	0.344902
4	0.8	2.047183	4	2.04718349	-3.79918E-13	0.344901
5	1		0	4.09436697	-1.305584697	-0.63775
5	1		1	2.04718349	-0.838192046	0.181127
5	1		2	6.67484184	-4.283044359	-1.66994
5	1		3	4.11004745	-1.315634047	-0.64402

5	1		4	2.06719765	-0.834647062	0.173121
5	1		5	6.88837678	-4.648753648	-1.75535
5	1		6	4.24004354	-1.402733793	-0.69602
5	1		7	2.22467186	-0.812344604	0.110131
5	1		8	9.60082128	-10.88151605	-2.84033

The calculations consisting of iterations and moving to the new time step for h = 0.1 are similar. The results from these calculations are shown in the table below. The computations were repeated with step sizes of 0.01 and 0.001 to get an idea of the effect of the step size on the error. These results are shown in the table below.

n	h	$y_{x = hn}$	Error(n)	Err(n)/Err(n/10)
10	0.1	2.775045	0.775045	
100	0.01	2.041443	0.041443	0.05347
1000	0.001	2.003988	0.003988	0.09623

For the two smallest step sizes, cutting the step size by a factor of 10 reduces the error by approximately the same factor. This is consistent with the global first-order error of the method. The ratio in reducing the step size from .1 to .100 is even smaller. These results show that the true operation of the order of the error is valid only for small step sizes, but not so small that we get roundoff error.

.n	t_n	y_n	m	$y_n^{(m)}$	$g(y_n^{(m)})$	$g'(y_n^{(m)})$
0	0	1				
1	0.1		0	1	-0.01	0.98
1	0.1		1	1.010204	-1E-06	0.979796
1	0.1		2	1.010205	-1.1E-14	0.979796
1	0.1	1.010205	3	1.010205	-1E-16	0.979796
2	0.2		0	1.02041	-0.01062	0.959184
2	0.2		1	1.031482	-2.5E-06	0.958741
2	0.2		2	1.031484	-1.3E-13	0.958741
2	0.2	1.031484	3	1.031484	-6.6E-17	0.958741
3	0.3		0	1.052764	-0.01197	0.936834
3	0.3		1	1.065541	-4.9E-06	0.936068
3	0.3		2	1.065546	-8.2E-13	0.936067
3	0.3	1.065546	3	1.065546	9.02E-17	0.936067
4	0.4		0	1.099608	-0.0143	0.912031
4	0.4		1	1.115291	-9.8E-06	0.910777
4	0.4		2	1.115302	-4.7E-12	0.910776
4	0.4	1.115302	3	1.115302	-8.3E-17	0.910776
5	0.5		0	1.165058	-0.01811	0.883494
5	0.5		1	1.185558	-2.1E-05	0.881444
5	0.5		2	1.185582	-2.8E-11	0.881442
5	0.5	1.185582	3	1.185582	0	0.881442
6	0.6		0	1.255862	-0.02435	0.849297
6	0.6		1	1.284535	-4.9E-05	0.845856
6	0.6		2	1.284593	-2E-10	0.845849
6	0.6	1.284593	3	1.284593	0	0.845849
7	0.7		0	1.383604	-0.03499	0.806295

7	0.7		1	1.427005	-0.00013	0.800219
7	0.7		2	1.42717	-1.9E-09	0.800196
7	0.7	1.42717	3	1.42717	0	0.800196
8	0.8		0	1.569747	-0.05455	0.748841
8	0.8		1	1.642595	-0.00042	0.737185
8	0.8		2	1.643171	-2.7E-08	0.737093
8	0.8	1.643171	3	1.643171	0	0.737093
9	0.9		0	1.859171	-0.09509	0.665349
9	0.9		1	2.002083	-0.00184	0.639625
9	0.9		2	2.004956	-7.4E-07	0.639108
9	0.9	2.004958	3	2.004958	-1.2E-13	0.639108
10	1		0	2.366745	-0.19836	0.526651
10	1		1	2.743391	-0.01419	0.451322
10	1		2	2.774823	-9.9E-05	0.445035
10	1	2.775045	3	2.775045	-4.9E-09	0.444991