

November 1 Homework Solutions

1. Shift the index in the power series $\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1}$ so that the power under the summation sign is x^m . Verify your result by writing the first four terms for each form of the series. Determine the range of x for which the solution will converge; this is known as the radius of convergence for the series.

To accomplish the shift, we define a new index $m = s - 1$ to get the term x^m in the summation sign. To do this we have to replace s by $m + 1$ everywhere in the summation. At the lower limit of $s = 2$, we have $m = 2 - 1 = 1$. Thus, the summation becomes

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1} = \sum_{m=1}^{\infty} \frac{(m+1)(m+1+1)}{(m+1)^2+1} x^{m+1-1} = \sum_{m=1}^{\infty} \frac{(m+1)(m+2)}{(m+1)^2+1} x^m$$

The first four terms in the original series (for all values of s from 2 to 5) are

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1} = \frac{6x}{5} + \frac{12x^2}{10} + \frac{20x^3}{17} + \frac{30x^4}{26} + \dots$$

The first four terms in the modified series (for all values of m from 1 to 4) are

$$\sum_{m=1}^{\infty} \frac{(m+1)(m+2)}{(m+1)^2+1} x^m = \frac{6x}{5} + \frac{12x^2}{10} + \frac{20x^3}{17} + \frac{30x^4}{26} + \dots$$

The first four terms are the same in both series.

The radius of convergence is given by equation (11) in section 5.1 (p. 172) of Kreyszig 10th edition.

$$R = \frac{1}{\lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|} = \frac{1}{\lim_{m \rightarrow \infty} \left| \frac{\frac{(m+2)(m+3)}{(m+2)^2+1}}{\frac{(m+1)(m+2)}{(m+1)^2+1}} \right|} = \frac{1}{\lim_{m \rightarrow \infty} \left| \frac{[(m+1)^2+1](m+3)}{[(m+2)^2+1](m+1)} \right|}$$

$$\lim_{m \rightarrow \infty} \left| \frac{[(m+1)^2+1](m+3)}{[(m+2)^2+1](m+1)} \right| = \lim_{m \rightarrow \infty} \left| \frac{m^3+5m^2+8m+6}{m^3+5m^2+9m+5} \right| = 1$$

The last step in taking this limit is the application of L'Hôpital's (old spelling L'Hospital's) Rule which states that the limit of a ratio where the numerator and denominator are both indefinite is

the ratio of the derivative of the numerator to the derivative of the denominator. This rule may be applied more than once to the same fraction. In our case we apply this rule two times to obtain the limit as m approaches infinity of the ratio $(6m + 5)/(6m + 5) = 1$. We thus conclude that the radius of convergence, which is 1 divided by the limit is $1/1$ or **$R = 1$** .

2. Find the two solutions for the differential equation $(x + 2)^2 y'' + (x + 2)y' - y = 0$, using the Frobenius method.

To simplify this task, we will use $z = x + 2$ as the independent variable. For this substitution, we will still have $d^2y/dz^2 = d^2y/dx^2$ and $dy/dz = dy/dx$. Thus, we seek the solution to $z^2 y'' + zy' - y = 0$ or $y'' + y'/z - y/z^2 = 0$.

We write the solution using Frobenius method in the following form.

$$y(z) = z^r \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+r} \quad [1]$$

Here the a_n are unknown coefficients and the value of r is also unknown. We can differentiate this series twice to obtain.

$$\frac{dy}{dz} = \sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1} \quad \text{and} \quad \frac{d^2y}{dz^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n z^{n+r-2} \quad [2]$$

We can substitute equations [1] and [2] into our differential equation: $y'' + y'/z - y/z^2$ to obtain the following equation.

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n z^{n+r-2} + \frac{1}{z} \left(\sum_{n=0}^{\infty} (n+r) a_n z^{n+r-1} \right) - \frac{1}{z^2} \left(\sum_{n=0}^{\infty} a_n z^{n+r} \right) = 0 \quad [3]$$

If we combine the z^2 and z terms outside the sums with the z terms inside the sums, each series sum has a term in z^{n+r-2} , and we can combine these into a single series as follows.

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) - 1] a_n z^{n+r-2} = \sum_{n=0}^{\infty} [(n+r)^2 - 1] a_n z^{n+r-2} = 0 \quad [4]$$

The first ($n = 0$) term in this series is $(r^2 - 1)a_0 x^{r-2}$. The indicial equation required to set the coefficient of $x^{r-2} = 0$, without setting $a_0 = 0$ is $r^2 = 1$. So the roots of the indicial equation are $r = \pm 1$. This is the case where the two roots differ by an integer. We proceed by examining the first solution where $r = 1$. From equation [4] we see that we have to satisfy the following equation for the coefficient of each power of x to vanish.

$$[(n+r)^2 - 1] a_n = [(n+1)^2 - 1] a_n = n(n+2) a_n = 0 \quad [5]$$

The only way to satisfy this equation for values of $n > 0$ is for $a_n = 0$ for $n > 0$. **Thus, the power series solution in equation [1] has only a single term.**

$$y(z) = \sum_{n=0}^{\infty} a_n z^{n+r} = \sum_{n=0}^{\infty} a_n z^{n+1} = a_0 z \quad [6]$$

In the case where the two roots of the indicial equation differ by an integer, the second solution is given by the following equation, where r_2 is the second root of the indicial equation and k may be zero.

$$y_2(z) = ky_1(z)\ln(z) + z^{r_2} \sum_{n=0}^{\infty} A_n z^n = ky_1(z)\ln(z) + \sum_{n=0}^{\infty} A_n z^{n+r_2} \quad [7]$$

If we differentiate this solution two times we obtain the following results.

$$\frac{dy_2(z)}{dz} = k \frac{dy_1(z)}{dz} \ln(z) + k \frac{y_1(z)}{z} + \sum_{n=0}^{\infty} A_n (n+r_2) z^{n+r_2-1} \quad [8]$$

$$\frac{dy_2^2(z)}{dz} = k \frac{dy_1^2(z)}{dz} \ln(z) + \frac{2k}{z} \frac{dy_1(z)}{dz} - \frac{ky_1(z)}{z^2} + \sum_{n=0}^{\infty} A_n (n+r_2-1)(n+r_2) z^{n+r_2-2} \quad [9]$$

If we substitute this result into our differential equation : $y'' + y/z - y/z^2 = 0$, we obtain

$$\begin{aligned} & k \frac{dy_1^2(z)}{dz} \ln(z) + \frac{2k}{z} \frac{dy_1(z)}{dz} - \frac{ky_1(z)}{z^2} + \sum_{n=0}^{\infty} A_n (n+r_2-1)(n+r_2) z^{n+r_2-2} + \\ & \frac{1}{z} \left[k \frac{dy_1(z)}{dz} \ln(z) + \frac{ky_1(z)}{z} + \sum_{n=0}^{\infty} A_n (n+r_2) z^{n+r_2-1} \right] - \frac{1}{z^2} \left[ky_1(z) \ln(z) + \sum_{n=0}^{\infty} A_n z^{n+r_2} \right] = 0 \end{aligned} \quad [10]$$

We can rearrange this equation, collecting all the terms multiplying $\ln(z)$ as follows.

$$\begin{aligned} & k \left[\frac{dy_1^2(z)}{dz} + \frac{1}{z} \frac{dy_1(z)}{dz} - \frac{1}{z^2} y_1(z) \right] \ln(z) + \frac{2k}{z} \frac{dy_1(z)}{dz} - \frac{ky_1(z)}{z^2} + \\ & \sum_{n=0}^{\infty} A_n (n+r_2-1)(n+r_2) z^{n+r_2-2} + \frac{1}{z} \left[k \frac{y_1(z)}{z} + \sum_{n=0}^{\infty} A_n (n+r_2) z^{n+r_2-1} \right] - \frac{1}{z^2} \sum_{n=0}^{\infty} A_n z^{n+r_2} = 0 \end{aligned} \quad [11]$$

The first term in brackets that multiplies $k \ln(z)$ is simply the original differential equation. Since y_1 is a solution to this equation, the term in brackets is zero. In addition there are two ky_1/z^2 terms with opposite signs that cancel. Eliminating these terms and combining the common summation terms gives the following equation.

$$\frac{2k}{z} \frac{dy_1(z)}{dz} + \sum_{n=0}^{\infty} [(n+r_2-1)(n+r_2) + (n+r_2)-1] A_n z^{n+r_2-2} = 0 \quad [12]$$

Since $y_1 = a_0 z$, $dy_1/dz = a_0$. Substituting $dy_1/dz = a_0$ and $r_2 = -1$ into equation 12 and simplifying the summation term gives.

$$\frac{2k}{z} a_0 + \sum_{n=0}^{\infty} [(n+r_2)^2 - 1] A_n z^{n+r_2-2} = \frac{2k}{z} a_0 + \sum_{n=0}^{\infty} n(n-2) A_n z^{n-3} = 0 \quad [13]$$

Writing the first few terms in this equation gives the following result.

$$0 - A_1 z^{-2} + 2ka_0 z^{-1} + 3A_3 + 8A_4 x + \sum_{n=0}^{\infty} n(n-2)A_n z^{n-3} = 0 \quad [14]$$

We must have $A_1 = k = A_3 = A_4 = 0$ to set the coefficients of the initial terms to zero. In addition, all subsequent values of A_n for $n > 4$ must be zero. Thus, the only possible nonzero values of A_n are A_0 and A_2 . Setting $k = 0$ and $r_2 = -1$ in equation

$$y_2(z) = ky_1(z) \ln(z) + \sum_{n=0}^{\infty} A_n z^{n+r_2} = A_0 z^{-1} + A_2 z \quad [15]$$

We see that this second solution contains the first solution, $y = a_0 z$. Thus we can use any linear combination of z and z^{-1} as a basis for the solutions to the original differential equation. Since we made an initial choice to solve our equation for $z = x + 2$ instead of x , we see that we can write our solutions in terms of the original x variable as $y_1 = x + 2$ and $y_2 = 1/(x + 2)$.

3. Use the equation from Kreyszig that $\frac{d}{dx} [x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$ to show that

$$\int x^{-\nu} J_{\nu+1}(x) dx = -x^{-\nu} J_{\nu}(x) + c.$$

Equation (25) on page 223 of Kreyszig states that $\frac{d}{dx} [x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$. If we multiply this equation by $-dx$ and take the indefinite integral of each side we obtain the result that $-\int d[x^{-\nu} J_{\nu}(x)] = \int x^{-\nu} J_{\nu+1}(x) dx + const$. Since $\int d[x^{-\nu} J_{\nu}(x)] = x^{-\nu} J_{\nu}(x)$, and the constant can be placed on any side of the equation, we have obtained the desired result.

4. Using the substitutions $y = u \sqrt{x}$ and $z = kx^2/2$, show that the differential equation $y'' + k^2 x^2 y = 0$ reduces to Bessel's equation. Find a general solution to the original equation in terms of Bessel functions.

From the definition of y we have $dy/dx = x^{1/2} du/dx + ux^{-1/2}/2$ and $d^2y/dx^2 = x^{1/2} d^2u/dx^2 - ux^{-3/2}/4 + x^{-1/2} du/dx$. Substituting this result into our original differential equation gives.

$$x^{1/2} \frac{d^2 u}{dx^2} + x^{-1/2} \frac{du}{dx} - x^{-3/2} \frac{u}{4} + k^2 x^2 u x^{1/2} = 0 \quad [1]$$

Multiplying this equation by $x^{3/2}$ gives

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} - \frac{u}{4} + k^2 x^2 u x^2 = 0 \quad [2]$$

If we shift the independent variable from x to $z = kx^2/2$, so that $x = (2z/k)^{1/2}$ and $dx/dz = (2/k)^{1/2} z^{-1/2} = (2kz)^{-1/2}$, we have

$$\frac{du}{dx} = \frac{dz}{dx} \frac{du}{dz} = (2kz)^{1/2} \frac{du}{dz} \quad [3]$$

$$\begin{aligned} \frac{d^2u}{dx^2} &= \frac{d}{dx} \left(\frac{du}{dx} \right) = \frac{dz}{dx} \frac{d}{dz} \left(\frac{du}{dx} \right) = \frac{dz}{dx} \frac{d}{dz} \left[(2kz)^{1/2} \frac{du}{dz} \right] = (2kz)^{1/2} \frac{d}{dz} \left[(2kz)^{1/2} \frac{du}{dz} \right] \\ &= (2kz)^{1/2} \left[(2kz)^{1/2} \frac{d^2u}{dz^2} + (2k)^{1/2} \frac{z^{-1/2}}{2} \frac{du}{dz} \right] = 2kz \frac{d^2u}{dz^2} + k \frac{du}{dz} \end{aligned} \quad [4]$$

Substituting $x = (2z/k)^{1/2}$ and the results of equations [3] and [4] into equation [2] gives.

$$\begin{aligned} \frac{2z}{k} \left(2kz \frac{d^2u}{dz^2} + k \frac{du}{dz} \right) + \left(\frac{2z}{k} \right)^{1/2} (2kz)^{1/2} \frac{du}{dz} - \frac{u}{4} + 4z^2u \\ = 4z^2 \frac{d^2u}{dz^2} + 2z \frac{du}{dz} + 2z \frac{du}{dz} - \frac{u}{4} + 4z^2u = 0 \end{aligned} \quad [5]$$

Combining the first derivative terms and dividing the equation by 4 gives

$$z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} + \left(z^2 - \frac{1}{16} \right) u = 0 \quad [6]$$

After dividing this equation by z^2 we see that this equation has the form of Bessel's equation,

$\frac{d^2u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \frac{z^2 - (1/4)^2}{z^2} u = 0$, with $\nu = 1/4$. Since this is a non-integer ν , we can write a general solution as $u = AJ_{1/4}(z) + BJ_{-1/4}(z)$. We are actually interested in a solution to our original equation with x as the independent variable and y as the dependent variable. Using the definitions of u and z we can obtain the desired final result as $y = x^{1/2} [AJ_{1/4}(kx^2/2) + BJ_{-1/4}(kx^2/2)]$.

5. The function $I_\nu(x) = i^{-\nu} J_\nu(ix)$, where $i^2 = -1$, is called the modified Bessel function of the first kind of order ν . Show that $I_\nu(x)$ is a solution of $x^2 y'' + xy' - (x^2 + \nu^2)y = 0$.

We can test this solution by plugging it into the differential equation. Because i is a constant, we can readily obtain the necessary derivatives of $y = I_\nu(x)$ to obtain the following result for the differential equation given in the problem statement.

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2)y = x^2 \frac{d^2[i^{-\nu} J_\nu(ix)]}{dx^2} + x \frac{d[i^{-\nu} J_\nu(ix)]}{dx} - (x^2 + \nu^2) i^{-\nu} J_\nu(ix) = 0$$

If we define a new variable, $z = ix$, so that $z^2 = -x^2$, and $dx = d(z/i)$, we can substitute this variable and bring the constant $i^{-\nu}$ outside the derivative operators to obtain.

$$(-z^2) i^{-\nu} \frac{d^2[J_\nu(z)]}{d(-z^2)} + \frac{z}{i} i^{-\nu} \frac{d[J_\nu(z)]}{d(z/i)} - (-z^2 + \nu^2) i^{-\nu} J_\nu(z) = 0$$

Cancelling common factors of i , and -1 in the second and first term and the common factor of $i^{-\nu}$ in all remaining terms gives.

$$z^2 \frac{d^2[J_\nu(z)]}{dz^2} + z \frac{d[J_\nu(z)]}{dz} + (z^2 - \nu^2)J_\nu(z) = 0$$

Dividing by z^2 gives Bessel's equation for which $J_\nu(z)$ is a solution.

$$\frac{d^2[J_\nu(z)]}{dz^2} + \frac{1}{z} \frac{d[J_\nu(z)]}{dz} + \frac{z^2 - \nu^2}{z^2} J_\nu(z) = 0$$

Thus, the proposed solution to the original differential equation is correct.

To obtain the power series solution for $I_\nu(x)$ we use the the power series equation for $J_\nu(x)$. We can manipulate the result using definition of $i^2 = -1$ to eliminate the $(-1)^m$ term in the numerator of the sum.

$$\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix) = i^{-\nu} (ix)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (ix)^{2m}}{2^{2m+\nu} m! \Gamma(m+\nu+1)} = \\ &= x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (i)^{2m} (x)^{2m}}{2^{2m+\nu} m! \Gamma(m+\nu+1)} = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(m+\nu+1)} = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu} m! \Gamma(m+\nu+1)} \end{aligned}$$

The final result is the series for $I_\nu(x)$ given in the problem statement.