October 25 Homework Solutions

1. Solve the equation y'' – 4y' = 0 by first converting it to a system of first-order equations and as given. Provide a general solution and give details of your work.

If we define z = y', the second order equation can be written as y'' = z' = 4y' = 4z. Thus, we have a system of two simultaneous first order equations

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} z \\ 4z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

The solution to this equation can be found by obtaining the eigenvalues of the matrix as follows.

$$Det(\mathbf{A} - \mathbf{I}\lambda) = \begin{bmatrix} -\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) = 0$$

The roots are $\lambda = 0$ and $\lambda = 4$. We next have to find the eigenvectors. For $\lambda = 0$ we have to solve the following equations for eigenvector components, x_1 and x_2.

$0x_1 + x_2 = 0$

$0x_1 + 4x_2 = 0$

This gives $x_2 = 0$ and $x_1 = a$, an undetermined constant. Thus the first eigenvector can be written as $a[1 \;\; 0]^T$. For $\lambda = 0$ we have to solve the following equations for eigenvector components, x_1 and x_2.

$-4x_1 + x_2 = 0$

$0x_1 + 0x_2 = 0$

These equations have the solution that $x_1 = b$ and $x_2 = 4b$, where b is undetermined; our second eigenvector can be written as $b[1 \;\; 4]^T$. Thus, we can write a general solution to the problem in terms of the eigenvectors and exponents of the eigenvalues as follows.

$$\begin{bmatrix} y \\ z \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^0 + b\begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{4t}$$

Since we were originally interested in the second order equation in y we see that the solution for y is $\boxed{y = a + be^{4t}}$ (Note that the solution for $z = 4be^{4t}$ equals y' as required.)

This equation has the general form y'' + ay' + by = 0, with a = -4 and b = 0. We solve the general equation by finding the roots of the characteristic equation. Here we find the roots as follows.

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(0)}}{2} = 4,0$$

Thus the general solution to our equation is $y = ae^{0t} + be^{4t}$ or **$y = a + be^{4t}$** as found previously.

2. Solve the following initial value problem showing all details: $y_1' = y_2 - 5 \sin t$ and $y_2' = -4y_1 + 17 \cos t$ with $y_1(0) = 5$ and $y_2(0) = 2$.

To solve this pair of equations we differentiate the first equation giving the following result:

$$y_1'' = y_2' - 5 \cos t$$

If we substitute the second equation for y_2' into this equation we get a second order differential equation that only contains y_1.

$$y_1'' = y_2' - 5 \cos t = -4y_1 + 17 \cos t - 5 \cos t = -4y_1 + 12 \cos t$$

The resulting equation, $y_1'' + 4y_1 = 12 \cos t$, is seen to have a homogenous equation, $y_1'' + 4y_1 = 0$, whose solution is A sin 2 t + B cos 2t. The complete solution to this equation requires the particular solution that we can find by the method of undetermined coefficients. We write the particular solution as $y_P = C \sin t + D \cos t$ so that $y_P'' = -C \sin t - D \cos t$. Substituting this particular solution and its derivatives into our differential equation gives the following result.

$$-C \sin t - D \cos t.+ 4(C \sin t + D \cos t.) = 12 \cos t$$

Equating the coefficients of the sine and cosine terms on each side of this equation gives the results that C = 0 and D = 4. Thus, $y_p = 4 \cos t$ and the complete solution for y_1, the sum of the homogenous and particular solution, is $y_1 = A \sin 2t + B \cos 2t + 4 \cos t$.

We have to obtain the solution for y_2 to apply the initial condition on that variable. According to the first equation, $y_2 = y_1' + 5 \sin t$. Differentiating the solution for y_1 that we just found gives the following result for y_2.

$$y_2 = y_1' + 5 \sin t. = 2A \cos 2t - 2B \sin 2t - 4 \sin t + 5 \sin t = 2A \cos 2t - 2B \sin 2t + \sin t$$

Applying the initial conditions on y_1 and y_2 gives the following results.

$$y_1(0) = 5 = A \sin 0 + B \cos 0 + 4 \cos 0 = B + 4 \quad => \quad B = 1$$

$$y_2(0) = 2 = 2A \cos 0 - 2B \sin 0 + \sin 0 = 2A \quad => \quad A = 1$$

Thus, our solutions are **$y_1 = \sin 2t + \cos 2t + 4 \cos t$** and

$y_2 = 2 \cos 2t - 2 \sin 2t + \sin t$

3. Find the power series solution of the differential equation y'' – 3y' + 2y = 0.

In the power series solution, we postulate a solution of the following form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad [1]$$

Here the a_n are unknown coefficients. We can differentiate this series twice to obtain.

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad and \qquad \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \qquad [2]$$

Substituting equations [1] and [2] into our original differential equation gives the following result.

$$\sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} - 3\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} + 2\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0 \qquad [3]$$

We want to rewrite the first two sums in this equation so that we have the same form as the final sum. We consider each of these separately. In the first sum of equation [3], the first two terms (for n = 0 and n = 1) are zero. Thus, we can write this sum as follows, where we introduce a new index, m = n-2.

$$\sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} (x - x_0)^m \qquad [4]$$

In a similar fashion, we note that the first (n = 0) term in the second sum is zero and we rewrite this sum using the new index m = n – 1 to give

$$\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = 0 + \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} (x - x_0)^m \qquad [5]$$

We now want to substitute equations [4] and [5] back into equation [3]. Before doing this we note that the summation index, n, in the final sum is a dummy index and re can replace this by m = n. This gives the following result in place of equation [3].

$$\sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} (x - x_0)^m - 3\sum_{m=0}^{\infty} (m+1) a_{m+1} (x - x_0)^m + 2\sum_{m=0}^{\infty} a_m (x - x_0)^m = 0 \qquad [6]$$

Since each sum in this equation has the same lower and upper limits we can combine them as follows.

$$\sum_{m=0}^{\infty} \left[(m+1)(m+2) a_{m+2} - 3(m+1) a_{m+1} + 2a_m\right](x - x_0)^m = 0 \qquad [7]$$

In order for this series sum to equal zero for any value of x, the coefficient of each power of x must equal zero. Thus we must satisfy the following equation for each value of m.

$$(m+1)(m+2)a_{m+2} - 3(m+1)a_{m+1} + 2a_m = 0 \quad [8]$$

We can solve this equation for a_{m+2}.

$$a_{m+2} = \frac{3(m+1)a_{m+1} - 2a_m}{(m+1)(m+2)} \quad [9]$$

We can use this equation to find any value of a_{m+2} in terms of previous coefficients a_{m+1} and a_m. Since we cannot solve this equation for a_0 and a_1, we will leave these constants to be used to satisfy boundary conditions on the differential equation. Setting m = 0 in equation [9], gives the value of a_2 as follows.

$$a_2 = a_{0+2} = \frac{3(0+1)a_{0+1} - 2a_0}{(0+1)(0+2)} = \frac{3a_1 - 2a_0}{2} \quad [10]$$

Setting m = 1 in equation [9] gives us the value of a_3 in terms of a_1 and a_2. We can use equation [10] to obtain an equation that then gives a_3 in terms of a_0 and a_1.

$$a_3 = a_{1+2} = \frac{3(1+1)a_{1+1} - 2a_1}{(1+1)(1+2)} = \frac{6a_2 - 2a_1}{6} = \frac{6\left(\frac{3a_1 - 2a_0}{2}\right) - 2a_1}{6} = \frac{7a_1 - 6a_0}{6} \quad [11]$$

We can continue in this fashion to try to deduce a general formula for a_m. Next we have m = 2.

$$a_4 = a_{2+2} = \frac{3(2+1)a_{2+1} - 2a_2}{(2+1)(2+2)} = \frac{9a_3 - 2a_2}{12} = \frac{9\left(\frac{7a_1 - 6a_0}{6}\right) - 2\left(\frac{3a_1 - 2a_0}{2}\right)}{12} = \frac{15a_1 - 14a_0}{24} \quad [12]$$

We do not appear to have a simple solution yet. The solution that we have so far, with terms up to and including $(x - x_0)^4$, can be written as follows.

$$y = a_0 + a_1(x - x_0) + \frac{3a_1 - 2a_0}{2}(x - x_0)^2 + \frac{7a_1 - 6a_0}{6}(x - x_0)^3 + \frac{15a_1 - 14a_0}{24}(x - x_0)^4 + \cdots$$

We see that the values of a_0 and a_1 are given by the initial conditions: $a_0 = y(x_0)$ and $a_1 = y'(x_0)$. Although we know that the differential equation has a simple solution in terms of exponentials, general power series solutions will not have simple expressions and we can just leave the solution in the form above, with the knowledge that we can get the coefficients for additional terms in the series by the recursion equation [9]. In a numerical application of a power series solution, we would not even need to reduce the individual coefficients by algebra to terms involving a_0 and a_1. Instead, we could use equation [9] to obtain numerical results for the coefficients.