College of Engineering and Computer Science
Mechanical Engineering Department

Mechanical Engineering 501A
Seminar in Engineering Analysis

Fall 2017 Number: 15966 Instructor: Larry Caretto

October 11 Homework Solutions

In addition to the usual problem solutions, there is an appendix with a very brief discussion of solving differential equations in MATLAB. Solutions to this week's problems using MATLAB are given in that appendix.

1. Verify that $y_p = 2x$ is a solution to the differential equation y'' +y = 2x. Solve the initial value problem for y(0) = -1 and y'(0) = 8.

Substituting the proposed solution into the differential equation gives the following result

$$\frac{d^2y}{dx^2} + y = 0 + 2x = 2x \quad so\ y_P = 2x\ is\ a\ solution$$

To solve the initial value problem, we have to obtain the homogenous solution. The general homogenous solution to this equation is y_H = A sin x + B cos x. So our general solution, y = y_H + y_P = A sin x + B cos x + 2x. At x = 0 we have y(0) = -1 = A sin 0 + B cos 0 + 2(0) = B so B = -1. Differentiating the solution and evaluating it at x = 0 gives y'(0) = 8 = A cos 0 – B sin 0 + 2 = A + 2. This gives A = 6 so that $\boxed{y = 6 \sin x - \cos x + 2x}$.is the solution to the initial value problem.

2. Solve the following initial value problem: y'' - 4y = e^{-2x} - 2x, y(0) = 0, y'(0) = 0. Indicate the rules you are using. Show each step of your calculation.

The solution to the homogenous equation, y = C e^{2x} + D e^{-2x} can be written by inspection. For the nonhomogenous solution, we have two different kinds of terms on the right hand side, an exponential and a first order polynomial. Thus we have to include particular solutions for each of these. We notice that the exponential term on the right-hand side of the differential equation is proportional to one of the homogenous solutions. Thus we use y_P = Axe^{-2x} + B + Cx for our particular solution here. Differentiating this solution two times and substituting it into our differential equation gives.

$$\frac{dy_P}{dx} = Ae^{-2x} - 2Axe^{-2x} + C \qquad \frac{d^2y_P}{dx^2} = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$$

$$\frac{d^2y_P}{dx^2} - 4y_P = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} - 4\left[Axe^{-2x} + B + Cx\right] = e^{-2x} - 2x$$

Equating coefficients of like terms on both sides of this equation gives the following results. For the x^0 terms, we have B = 0. For the x terms we have -4C = -2; this gives C = ½. For the e^{-2x} terms we have -2A -2A = 1 so that A = -1/4. Finally, we see that the coefficient of the xe^{-2x} terms on the left side of the equation is 4A – 4A = 0, matching the zero coefficient for this term on the left-hand side. Based on these values for A, B, and C, our particular solution is $y_P = x/2 – xe^{-2x}/4$. Thus our general solution to the differential equation is $y = C e^{2x} + D e^{-2x} + x/2 – xe^{-2x}/4$.

To solve the initial value problem we have to use the initial conditions, y(0) = 0 and y'(0) = 0 to evaluate the constants C and D. At x = 0 we have $y = Ce^0 + De^0 +(0)/2 – (0)e^0/4 = C + D$. Thus, our first boundary condition tells us that C = -D. To get the second boundary condition we differentiate our solution giving $y' = 2Ce^{2x} – 2De^{-2x} + ½ - e^{-2x}/4 -+2xe^{-2x}/4$. The initial condition that y'(0) = 0 gives the following equation: $y'(0) = 0 = 2Cd^0 – 2De^0 + ½ - e^0/4 + 2(0)e^0/4$. We now have two equations for C and D. The second equation tells us that 2C – 2D = ¼ - ½ = -1/4. If we substitute the first equation, C = -D, into this first equation we obtain -2D -2D = -4D = -1/4 or D = 1/16. Since C = -D, we have C = -1/16. Thus the solution to our initial value problem is the following: $\boxed{y = (e^{-2x} – e^{2x})/16 + x/2 – xe^{-2x}/4}$.

We can verify this solution by substituting it into the differential equation; we can also write the term $e^{-2x} – e^{2x}$ as –2sinh(2x).

3. Find the motion of the spring-mass system given by the differential equation y'' + 25y = 24 sin t, with the initial conditions y(0) = 1 and y'(0) = 1. Sketch or plot the solution curve. State the time when the solution practically reaches the steady state. (Show the details of your work.)

The solution to the homogenous equation can be written by inspection as y_H = C sin 5t + D cos 5t. For the right-hand side, r(t) = 24 sin t, we use the following trial function for the particular solution: y_P = A sin t + B cos t. The second derivative of this solution is d^2y_P/dt^2 = -A sin t – B cos t. Substituting the trial particular solution into the differential equation gives the following result.

$$\frac{d^2y_P}{dt^2} + 25y_P = -A\sin t - B\cos t + 25(A\sin t + B\cos t) = 24\sin t$$

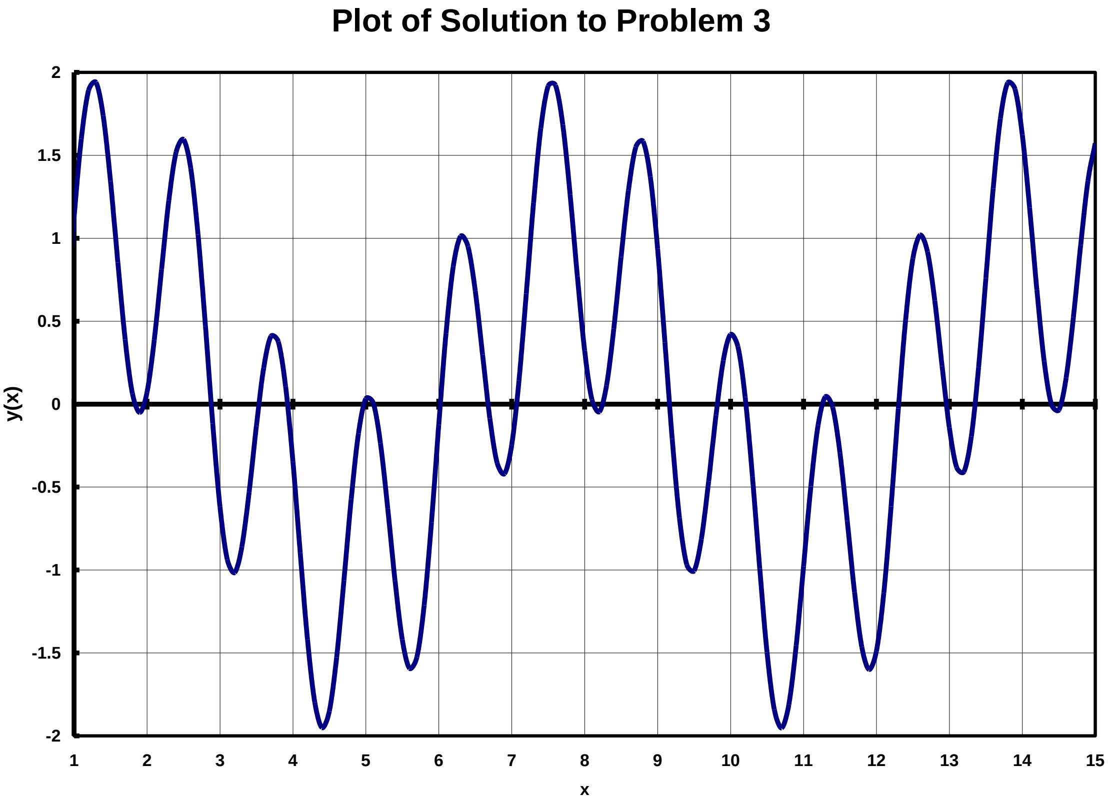
We see that the coefficients of the sine terms give us –A + 25A = 24 or A = 1. The cosine terms give the following result: -B + 25B = 0 or B = 0. Thus, our particular solution is y_P = sin t. Our solution for the differential equation then becomes $y = y_H + y_P$ = C sin 5t + D cos 5t + sin t. For y(0) = 1 we have y(0) = 1 = C sin 0 + D cos 0 + sin 0 = D or D = 1. For the initial condition that y'(0) = 1, we have the following result: $y'(0) = 1 = [5C \cos 5t – 5D \sin 5t + \cos t]_{t=0} = 5C + 1$. This equation gives C = 0. Thus the solution to the initial value problem is $\boxed{y = \cos 5t + \sin t}$.

Since this solution does not have any exponential or polynomial terms, it is at a steady state for all times. The two components of the solution, the relatively high frequency cosine term and the lower frequency sine term can be clearly distinguished on the plot of the solution that is shown on the next page.

4. Solve the initial value problem y''' + 3y'' + 3y' + y = e^{-x}sin x, with y(0) = 2, y'(0) =0, and y''(0) = -1. Show your work.

We start by finding the solution to the homogenous equation. To do this we have to find the roots of the equation $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$. The coefficients are just the expansion coefficients for $(\lambda + 1)^3$. So we have a triple root here with λ = -1. The three linearly independent solutions are e^{-x}, xe^{-x}, and x^2e^{-x}.

To get a trial particular solution with a right-hand side of e^{-x}sin x, we use a trial solution of e^{-x}[A sin x + B cos x]. Although the e^{-x} term on the right-hand side looks like the e^{-x} term in the homogenous solution, the multiplication of this exponential by sin x makes it linearly independent from the homogenous solution. The first three derivatives of this trial solution are shown below the plot of the solutions to the previous problem.



$$\frac{dy_P}{dx} = -e^{-x}\left[A\sin x + B\cos x\right] + e^{-x}\left[A\cos x - B\sin x\right]$$

$$\frac{d^2y_P}{dx^2} = e^{-x}\left[A\sin x + B\cos x\right] - 2e^{-x}\left[A\cos x - B\sin x\right] + e^{-x}\left[-A\sin x - B\cos x\right]$$
$$= -2e^{-x}\left[A\cos x - B\sin x\right]$$

$$\frac{d^3y_P}{dx^3} = 2e^{-x}\left[A\cos x - B\sin x\right] - 2e^{-x}\left[-A\sin x - B\cos x\right]$$

Substituting these derivatives and the original solution into the differential equation gives

$$\frac{d^3y_P}{dx^3} + 3\frac{d^2y_P}{dx^2} + 3\frac{dy_P}{dx} + y_P = 2e^{-x}\left[A\cos x - B\sin x\right] - 2e^{-x}\left[-A\sin x - B\cos x\right] - 6e^{-x}[A\cos x$$
$$-B\sin x] - 3e^{-x}\left[A\sin x + B\cos x\right] + 3e^{-x}\left[A\cos x - B\sin x\right] + e^{-x}\left[A\sin x + B\cos x\right] = e^{-x}\sin x$$

Since the e^{-x} factor is common to all terms, we can cancel it. We then set the coefficients of the sine terms on the left equal to 1 and the cosine terms equal to zero to match the presence of these terms on the right-hand side. Doing this for the sine terms gives

$$2[-B\sin x]-2[-A\sin x]-6B\sin x-3[A\sin x]+3[-B\sin x]+[A\sin x]=\sin x$$

$$B(-2+6-3)+A(2-3+1)=1 \qquad \Rightarrow \qquad B=1$$

The cosine terms give

$$2e[A\cos x]-2[-B\cos x]-6[A\cos x]-3[B\cos x]+3[A\cos x]+[B\cos x]=0$$

$$B(2-3+1)+A(2-6+3)=0 \qquad \Rightarrow \qquad -A=0$$

Thus, our particular solution is simply $y_P = e^{-x}\cos x$. Adding this to the homogenous solution gives the general solution as $y = (C_1 + C_2x + C_3x^2)e^{-x} + e^{-x}\cos x$. The first two derivatives of this solution are shown below.

$$\frac{dy}{dx}=(C_1+C_2x+C_3x^2)(-e^{-x})+(C_2+2C_3x)e^{-x}-e^{-x}\cos x+e^{-x}[-\sin x]$$

$$\frac{d^2y}{dx^2}=(C_1+C_2x+C_3x^2)(e^{-x})-2(C_2+2C_3x)e^{-x}+2C_3e^{-x}+e^{-x}\cos x-2e^{-x}[-\sin x]-e^{-x}\cos x$$

Substituting the initial conditions that y(0) = 2, y'(0) =0, and y''(0) = -1 gives the following results.

$$y(0) = 2 = (C_1 + C_2(0) + C_3(0)]e^0 + e^0\cos(0) = C_1 + 1 \text{ so that } C_1 = 1$$

$$\left.\frac{dy}{dx}\right|_{y=0}=0=[C_1+C_2(0)+C_3(0)](-e^0)+[C_2+2C_3(0)]e^0-e^0\cos 0+e^0[-\sin 0]$$

$$\left.\frac{dy}{dx}\right|_{y=0}=0=-C_1+C_2-1 \qquad \Rightarrow C_2=1+C_1=2$$

$$\left.\frac{d^2y}{dx^2}\right|_{y=0}=[1+C_2(0)+C_3(0)]e^0-2[C_2+2C_3(0)]e^0+2C_3e^0+e^0\cos 0-2e^0[-\sin 0]-e^0\cos 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{y=0}=-1=C_1-2C_2+2C_3+1-1 \qquad \Rightarrow \qquad C_3=\frac{-1+2C_2-C_1}{2}=\frac{-1+4-1}{2}=2$$

With these constants our solution is $\boxed{y = (1 +2x+ x^2)e^{-x}\ \ e^{-x}\cos x}$.

APPENDIX – USING MATLAB

MATLAB uses the command `dsolve` to solve differential equations. The equation and the initial conditions are entered as arguments to this function. By default, the independent variable is t and the dependent variable is y. The symbol D is used to indicate a derivative. For example, Dy means dy/dt; D2y means d^2y/dt^2 and so forth. The differential equation $d^2y/dx^2 + 3\ dy/dx + 2y = e^x \cos x$ is specified in MTALAB as the following formula: 'D2y + 3 * Dy + 2 * y = exp(x) * cos(x)'. Note the use of the single quotation marks.

The `dsolve` function can be called with a single argument giving the differential equation. In this case a general solution is returned with constants that have to be evaluated. You can also place initial and boundary conditions in the arguments to `dsolve`. Such conditions are specified by the notation y(a) = b to indicate that y has a value of b at t = a. For example the initial condition that y is 3 at t = 0 would be indicated as 'y(0) = 3'. Initial or boundary conditions on derivatives are indicated using the D notation. An initial condition that dy/dt = 1 at t = 0 would be entered as 'Dy(0) = 1'.

Solving the differential equation $d^2y/dx^2 + 3\ dy/dx + 2y = e^x \cos x$ with initial conditions that y(0) = 3 and dy/dt = 1 at t = 0 would be done by the following command:

```
dsolve('D2y + 3 * Dy + 2 * y = exp(x) * cos(x)', 'y(0) = 3', 'Dy(0) = 1')
```

MATLAB produces the following answer in response to this `dsolve` command.

```
1/2*exp(x)*cos(x)-exp(-2*t)*(-1/2*cos(x)*cosh(x)-
1/2*cos(x)*sinh(x)+4)+exp(-t)*(-cos(x)*cosh(x)-cos(x)*sinh(x)+7)
```

The following text was copied from a MATLAB session to solve the problems on this homework assignment.

```
>> dsolve('D2y+y=2*t','y(0)=-1','Dy(0)=8')   %Problem 1
ans =
6*sin(t)-cos(t)+2*t

>>%Problem 2
>> dsolve('D2y-4*y=exp(-2*t)-2*t','y(0)=0','Dy(0)=0')
ans =
-1/16*exp(2*t)+1/8*exp(-2*t)+1/16*(-1+8*t*exp(2*t)-4*t)*exp(-2*t)

>>%Problem 3
>> dsolve('D2y+25*y=24*sin(t)','y(0)=1','Dy(0)=1')

ans =
cos(5*t)+sin(t)

%Create plot shown for problem 3 in solutions above
>> t = 0:0.1:20;   %Set t values from 0 to 20 in increments of 0.1
>> plot(t,y)       %Plot y = cos(5*t)+sin(t) for t set in line above

EDU>>%Problem 4
EDU>> dsolve('D3y+3*D2y+3*Dy+y=exp(-t)*sin(t)','y(0)=2','Dy(0)=0','D2y(0)=-1')

ans =
exp(-t)*cos(t)+exp(-t)+exp(-t)*t^2+2*exp(-t)*t
```