California State University Northridge

College of Engineering and Computer Science
Mechanical Engineering Department

Mechanical Engineering 501A
Seminar in Engineering Analysis

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October 4 Homework Solutions

1. Solve the following initial value problem: y' + 4y = 20, with y(0) = 2. Show the details of your work.

This is an example of the general first order linear equation: y' + f(x)y = g(x) with f(x) = 4 and g(x) = 20. The solution to this equation, for the general case can be adapted to the problem considered here as shown below.

$$y = e^{-\int f(x)dx}\left[C + \int e^{\int f(x)dx} g(x)dx\right] \quad \Rightarrow \quad y = e^{-\int 4dx}\left[C + \int e^{\int 4dx} 20dx\right]$$

Performing the indicated integrations gives our solution with the general constant, C.

$$y = e^{-4x}\left[C + \int e^{4x} 20dx\right] = e^{-4x}\left[C + \frac{20}{4}e^{4x}\right] = Ce^{-4x} + 5$$

We select C to meet the initial condition that y(0) = 2. This gives y(0) = 2 = Ce^0 + 5 = C + 5, so C = -3 and our solution is $\boxed{y = 5 - 3e^{-4x}}$.

We can verify this solution by substituting it into the differential equation. Since dy/dx = 12e^{-4x}, this substitution gives y' +4y = 12e^{-4x} +4(5 – 3e^{-4x}) = 20 which matches the right-hand side. Also, y(0) = 5 – 3e^0 = 2, matching the initial condition.

2. Solve the following initial value problem: y' = 1 + y^2, with y(0) = 0. Show the details of your work.

This is an example of a separable first order differential equation. Writing y' as dy/dx and multiplying both sides of the equation by (1 + y^2) allows us to perform an indefinite integration on each side.

$$\frac{dy}{dx} = 1 + y^2 \quad \Rightarrow \quad \int \frac{dy}{1+y^2} = \int dx + C \quad \Rightarrow \quad \tan^{-1}(y) = x + C$$

Since $\tan^{-1}(0)$ = 0, the initial condition gives C = 0 and our solution is $\tan^{-1}(y)$ = x or $\boxed{y = \tan(x)}$.

3. Show that the initial value problem xy' = 4y, with y(0) = 1 has no solution. Does this contradict our present existence theorem?

The equation xy' = 4y is another example of a separable first order differential equation. Writing y' as dy/dx and multiplying both sides of the equation by dx/(xy) allows us to perform an indefinite integration on each side.

$$x\frac{dy}{dx} = 4y \qquad \Rightarrow \qquad \int\frac{dy}{y} = 4\int\frac{dx}{x} \qquad \Rightarrow \qquad \ln(y) = \ln(x) + C$$

We cannot apply the initial condition that y(0) = 1 to the ln(x) term. Thus, we cannot find a solution to this differential equation with the given initial condition. This is consistent with the existence theorem which requires the derivative, f(x,y), in the differential equation dy/dx = f(x,y) to be bounded in the region of the solution. At the proposed initial condition, dy/dx = 4y/x is indefinite and therefore not bounded. Thus, we do not expect a solution to exist for the given differential equation and initial condition.

4. Find all initial conditions such that the initial value problem (x^2 – 2x)y' =2(x – 1)y, with y(x_0) = y_0 has (a) no solution, (b) more than one solution, and (c) precisely one solution.

If we rewrite this differential equation in the form used in the existence and uniqueness theorems we have the following result

$$\frac{dy}{dx} = f(x,y) = \frac{2(x-1)y}{x^2-2x} \qquad \Rightarrow \qquad \frac{\partial f}{\partial y} = \frac{2(x-1)}{x^2-2x}$$

We see that the value of f(x,y) is not bounded at x = 0 and x = 2. Thus, there are no solutions at these points. The value of ∂f/∂y is bounded at all points except the same points where there is no solution. Thus, it does not appear that there are any regions where a solution exists and is not unique.

We can check this by trying to solve the differential equation, which has a separable solution. Rearranging the original equation gives the following result.

$$\frac{dy}{y} = \frac{2(x-1)}{x^2-2x} \qquad \Rightarrow \qquad \ln y = \int\frac{2(x-1)}{x^2-2x}dx + C = \int\frac{2x}{x^2-2x}dx - \int\frac{2}{x^2-2x}dx + C$$

We can evaluate the integrals on the left side of the solution for ln y as follows.

$$\int\frac{2x}{x^2-2x}dx = \int\frac{2}{x-2}dx = 2\ln(x-2)$$

$$\int\frac{2}{x^2-2x}dx = \int\left[\frac{1}{x-2} - \frac{1}{x}\right]dx = \ln(x-2) - \ln x$$

Substituting these integrals into the equation for ln y gives the following result.

$$\ln y = \int\frac{2x}{x^2-2x}dx - \int\frac{2}{x^2-2x}dx + C = 2\ln(x-2) - \ln(x-2) + \ln x + C = \ln(x-2) + \ln x + C$$

Taking the exponential of both sides of this equation and rearranging gives

$$y = e^{\ln y} = e^{\ln(x-2)+\ln x + C} = e^{\ln(x-2)}e^{\ln x}e^{C} = (x-2)xC$$

We can check this solution by differentiating it and substituting it into our original differential equation.

$$\frac{dy}{dx} = C(x-2) + Cx = 2C(x-1) \qquad \Rightarrow \qquad \frac{1}{y}\frac{dy}{dx} = \frac{2C(x-1)}{Cx(x-2)} = \frac{2(x-1)}{(x^2-2x)}$$

The final result above matches our differential equation showing that we have the correct solution. Now let's see what we can do with this solution, y = Cx(x-2). If x = 0 or x = 2, the only solution we have is the trivial one that y = 0. We cannot fit an arbitrary boundary condition $y(x_0) = y_0$ at these points. At any other points, the initial condition that $y(x_0) = y_0$ gives $C = y_0/x_0/(x_0 - 2)$. So our solution is $y = y_0x(x - 2)/x_0/(x_0 - 2)$.

5. Verify that the functions cos 3x and sin 3x form a basis of solutions for the differential equation y'' + 9y = 0 and solve the initial value problem with y(0) = 4 and y'(0) = -6.

Differentiating the proposed solutions two times and substituting the results for y and y'' into the original differential equation shows that both solutions satisfy the differential equation: -9 sin 3x + 9 sin 3x = 0; -9 cos 3x + 9 cos 3x = 0. Since the sine and cosine are two linearly independent functions, they form a basis for the second-order differential equation.

To match the given initial conditions, we use a linear combination of the two solutions to write y = A sin 3x + B cos 3x. The first initial condition gives y(0) = 4 = A sin 0 + B cos 0 = B, so B = 4. The second initial condition gives y'(0) = 3A cos 0 – 3B sin 0 = 3A = -6 so A = -2. Thus, the solution is **y = -2 sin 3x + 4 cos 3x**

6. Solve the following initial value problem: 9y'' + 6y' + y = 0; y(0) = 4; y'(0) = -13/3. (Show each step.)

The solution to the differential equation y'' + ay' + by = 0 depends on the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$. For this problem, we have to divide the given differential equation by the y'' coefficient to obtain the following result y'' + 2y'/3 + y/9 = 0. The solution to the characteristic equation with a = 2/3 and b = 1/9 is found below.

$$\lambda = \frac{-\frac{2}{3} \pm \sqrt{\left(\frac{2}{3}\right)^2 - 4(1)\frac{1}{9}}}{(2)(1)} = -\frac{1}{3}$$

Here the characteristic equation has a double root of –1/3, so the general solution to the differential equation is $y = (C_1 + C_2x)e^{-x/3}$. Fitting the first initial condition, y(0) = 4 gives $y(0) = 4 = (C_1 + 0\,C_2)e^0 = C_1$, so $C_1 = 4$. The second initial condition requires the derivative of our solution. Setting $C_1 = 4$ and taking the first derivative gives

$$y' = (4 + C_2x)\left(-\frac{1}{3}\right)e^{-x/3} + C_2e^{-x/3} \quad \Rightarrow \quad y'(0) = -\frac{13}{3} = (4 + 0C_2)\left(-\frac{1}{3}\right)e^0 + C_2e^0 = -\frac{4}{3} + C_2$$

The initial condition for y'(0) gives $C_2 = -3$ so our solution is $\boxed{y = (4 - 3x)e^{-x/3}}$.

7. Solve the following boundary value problem: y'' + 2y' + 2y = 0; y(0) = 1; y(π/2) = 0. (Show each step.)

The solution to the differential equation y'' + ay' + by = 0 depends on the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$. For this problem we have a = b =2. The solution for this characteristic equation is found below.

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)2}}{(2)(1)} = -1 \pm i$$

Here the characteristic equation has a complex roots so the general solution to the differential equation is $y = e^{-x}(A \sin x + B \cos x)$. Fitting the first boundary condition, y(0) = 1 gives $y(0) = 1 = e^{-0}(A \sin 0 + B \cos 0) = B$, so B = 1. The second boundary condition that y(π/2) = 0 gives $y(\pi/2) = 0 = e^{-\pi/2}(A \sin \pi/2 + B \cos \pi/2) = e^{-\pi/2}$, This boundary condition gives A = 0. Thus, our solution is $\boxed{y = e^{-x}\cos x}$.

8. Find a second-order homogenous linear differential equation for which the functions e^{3x} and xe^{3x} are solutions. Find the Wronskian and use it to verify that these solutions are independent.

We recognize there two solutions as solutions to $\frac{d^2y}{dt^2} + a\frac{dy}{dy} + by = 0$ in the case of the double root where $b = a^2/4$ and the exponent of the exponential is –ax/2. In this case where the exponent is 3, we must have a = -6 and $b = a^2/4 = 36/4 = 9$. Thus the differential equation is . $\frac{d^2y}{dt^2} - 6\frac{dy}{dy} + 9y = 0$. We can verify this result by substituting a general solution, $y = (A + Bx)e^{3x}$ into the differential equation. This gives

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dy} + 3y = 9(A + Bx)e^{3x} + 2B(3e^{3x}) - 6\left[(A + Bx)3e^{3x} + Be^{3x}\right] + 9(A + Bx)e^{3x}$$
$$= Ae^{3x}\left[9 - 18 + 9\right] + Be^{3x}\left[9x + 6 - 18x - 6 + 9x\right] = 0 \qquad (okay!)$$

The Wronskian for these two solutions is shown below.

$$\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{3x}\left(e^{3x} + 3xe^{3x}\right) - 3e^{3x}\left(xe^{3x}\right) = e^{3x}\left(e^{3x}\right) = e^{6x}$$

Since e^{6x} cannot be zero, the Wronskian for these two solutions is non-zero, showing the linearly independence of the two solutions.