College of Engineering and Computer Science
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Mechanical Engineering 501A
Seminar in Engineering Analysis

Fall 2017 Number 15966 Instructor: Larry Caretto

September 27 Homework Solutions

1. **Find the inverse of the matrix** A **at the right using Gauss-Jordan elimination. Check your result by showing that** AA^{-1} = I.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

In the Gauss-Jordan procedure we first augment the original matrix with a unit matrix as shown below.

$\begin{bmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$	In Gauss Jordan, we treat each row as a pivot row. We first divide the row by the pivot element then subtract the pivot row (times a factor) from all other rows so as to result in a unit matrix in place of the original matrix. We start by dividing the first row by two and subtracting 5 times the result from the second row. The result is shown below.
$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 2.5 & -2.5 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$	We do not have to divide by a_{22}, the pivot element in row 2, because it is already 1. Since $a_{12} = 0$ we do not have to subtract the pivot row from the first row. The only operation required is the subtraction of (one times) the pivot row from row three. This produces the result shown below.
$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 2.5 & -2.5 & 1 & 0 \\ 0 & 0 & 0.5 & 2.5 & -1 & 1 \end{bmatrix}$	For the final step, we divide the third row by $a_{33} = 0.5$. this gives [0 0 1 5 –2 2] for the final row. We then subtract -0.5 times the result from row one and 2.5 times the result from row two. These subtractions give the matrix below.
$\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{bmatrix}$	Our operations are now complete. We have the unit matrix in place of the original matrix and we have the inverse matrix in the right three columns. We can check our result by seeing if $\mathbf{AA}^{-1} = \mathbf{I}$.

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 2(3)+0(-15)+(-1)(5) & 2(-1)+0(6)+(-1)(-2) & 2(1)+0(-5)+(-1)(2) \\ 5(3)+1(-15)+(0)(5) & 5(-1)+1(6)+(0)(-2) & 5(1)+1(-5)+(0)(2) \\ 0(3)+1(-15)+3(5) & 0(-1)+1(6)+3(-2) & 0(1)+1(-5)+3(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

2. Find the inverse of the matrix A **at the right using the cofactor formula. Check your result by showing that** $A^{-1}A = I$**.**

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

If $\mathbf{B} = \mathbf{A}^{-1}$, then $b_{ij} = C_{ji}/\text{Det }\mathbf{A}$, where Cij is the cofactor of a_{ij}.

We start by obtaining the determinant of the original matrix, **A**.

$$Det\ \mathbf{A} = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} \begin{aligned} &= (2)(1)(3) + (5)(1)(-1) + (0)(0)(0) \\ &- (0)(1)(-1) - (5)(0)(3) - (2)(1)(0) \\ &= 6 - 5 + 0 - 0 - 0 - 0 = \mathbf{1} \end{aligned}$$

We use this value to compute the components of the inverse, b_{ij}, one by one. Note that the cofactor $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the "minor" determinant formed when row i and column j are removed from the original determinant. (Note the index reversal in the formula for the inverse; we use C_{ji} to compute b_{ij}.) We start with the upper left, b_{11}, and proceed by columns, computing all the elements in one column before proceeding to the next one.

$$b_{11} = \frac{(-1)^{1+1}M_{11}}{Det\ \mathbf{A}} = \frac{1}{1}\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = 1[(1)(3) - (1)(0)] = 3$$

$$b_{21} = \frac{(-1)^{2+1}M_{12}}{Det\ \mathbf{A}} = -\frac{1}{1}\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = -1[(5)(3) - (0)(0)] = -15$$

$$b_{31} = \frac{(-1)^{3+1}M_{13}}{Det\ \mathbf{A}} = \frac{1}{1}\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 1[(5)(1) - (0)(1)] = 5$$

$$b_{12} = \frac{(-1)^{1+2}M_{21}}{Det\ \mathbf{A}} = -\frac{1}{1}\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} = -1[(0)(3) - (1)(-1)] = -1$$

$$b_{22} = \frac{(-1)^{2+2}M_{22}}{Det\ \mathbf{A}} = \frac{1}{1}\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 1[(2)(3) - (0)(-1)] = 6$$

$$b_{32} = \frac{(-1)^{3+2}M_{23}}{Det\ \mathbf{A}} = -\frac{1}{1}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -1[(2)(1) - (0)(0)] = -2$$

$$b_{13} = \frac{(-1)^{1+3}M_{31}}{Det\ \mathbf{A}} = \frac{1}{1}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1[(0)(0) - (1)(-1)] = 1$$

$$b_{23} = \frac{(-1)^{2+3}M_{32}}{Det\ \mathbf{A}} = -\frac{1}{1}\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = -1[(2)(0) - (5)(-1)] = -5$$

$$b_{33} = \frac{(-1)^{3+3}M_{33}}{Det\ \mathbf{A}} = \frac{1}{1}\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = 1[(2)(1) - (5)(0)] = 2$$

We see that these elements are the same as the ones found using the Gauss-Jordan procedure in problem 1. A check is really not necessary here, but we can redo the check done in the first problem, using a reverse order to show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ in this case.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 3(2)+(-1)(5)+1(0) & 3(0)+(-1)(1)+1(1) & 3(-1)+(-1)(0)+1(3) \\ (-15)(2)+6(5)+(-5)(0) & (-15)(0)+6(1)+(-5)(1) & (-15)(-1)+6(0)+(-5)(3) \\ 5(2)+(-2)(5)+2(0) & 5(0)+(-2)(1)+2(1) & 5(-1)+(-2)(0)+2(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Use the Visual Basic code and data provided to obtain solutions to a set of 100 simultaneous linear equations. Obtain these results for each of 12 given right-hand sides. Compare solutions using single and double precision with and without pivoting.

The solutions are shown in the table below. Using pivoting decreases the average relative error by a factor of about 0.03 for both single and double precision. However, changing calculations from single to double precision reduces the average relative error by a factor of about 2x10^{-9} for both types of calculations: with and without pivoting. For the particular system of equations used here, the average relative error, with single precision and pivoting is 63.7x10^{-7} which is about 60 times larger than the smallest possible relative error in any calculation. Thus, the Gauss elimination method, even with pivoting, leads to significant roundoff errors with a system of equations of this size.

You can download the spreadsheet that was used to obtain these results from the course web site: http://www.csun.edu/~lcaretto/me501a/GaussSolution.xls.

The condition number of the 100 x 100 **A** matrix in this problem is 565 from the Matlab function cond. Thus, this matrix is not only relatively large; it is also ill-conditioned.

Summary of RMS Error Results for Each Right-hand Side Using Excel VBA Code								
Solution index (or average)	Double Precision Errors				Single Precision Errors			
	With Pivoting		Without Pivoting		With Pivoting		Without Pivoting	
	Absolute	Relative	Absolute	Relative	Absolute	Relative	Absolute	Relative
RHS 1	4.05E-13	6.70E-15	3.14E-11	5.20E-13	6.15E-04	1.02E-05	1.41E-02	2.33E-04
RHS 2	6.44E-13	1.17E-14	1.85E-11	3.36E-13	4.21E-04	7.63E-06	1.58E-02	2.87E-04
RHS 3	8.24E-13	1.45E-14	2.63E-11	4.63E-13	3.62E-04	6.36E-06	1.34E-02	2.36E-04
RHS 4	1.07E-12	1.77E-14	2.47E-11	4.12E-13	6.03E-04	1.00E-05	1.39E-02	2.31E-04
RHS 5	9.39E-13	1.65E-14	1.76E-11	3.09E-13	2.99E-04	5.26E-06	8.62E-03	1.52E-04
RHS 6	7.91E-13	1.28E-14	1.79E-11	2.89E-13	3.91E-04	6.32E-06	1.73E-02	2.80E-04
RHS 7	8.53E-13	1.53E-14	3.24E-11	5.82E-13	1.99E-04	3.56E-06	1.08E-02	1.94E-04
RHS 8	4.96E-13	8.89E-15	3.41E-11	6.11E-13	2.64E-04	4.72E-06	1.39E-02	2.49E-04
RHS 9	7.84E-13	1.36E-14	2.77E-11	4.82E-13	3.19E-04	5.54E-06	7.85E-03	1.36E-04
RHS 10	4.56E-13	7.50E-15	3.20E-11	5.27E-13	4.49E-04	7.39E-06	1.48E-02	2.43E-04
RHS 11	5.58E-13	9.13E-15	3.35E-11	5.49E-13	3.84E-04	6.29E-06	1.48E-02	2.43E-04
RHS 12	6.72E-13	1.07E-14	1.91E-11	3.05E-13	1.85E-04	2.96E-06	1.21E-02	1.93E-04
Average	7.07E-13	1.20E-14	2.63E-11	4.47E-13	3.74E-04	6.37E-06	1.31E-02	2.23E-04

The table on the next page shows the absolution and relative errors for two other applications: (1) using Excel matrix inversion followed by multiplication of the inverse **A** matrix with the right hand side matrix, **b**, and (2) using the MATLAB commands shown on slide 39 of the September 20 lecture to read **A** and **b** data from the excel file, solve for the solutions, **x**, and compute the RMS errors in the solution.

RMS Errors from Excel and MATLAB Solutions				
Right-hand	Excel Matinv		MATLAB	
Side	Absolute	Relative	Absolute	Relative
1	6.98E-12	1.38E-13	1.95E-14	3.85E-16
2	5.80E-12	1.30E-13	2.62E-15	5.85E-17
3	6.18E-12	1.32E-13	1.92E-14	4.10E-16
4	6.77E-12	1.30E-13	1.03E-14	1.98E-16
5	5.75E-12	1.34E-13	1.26E-14	2.94E-16
6	6.78E-12	1.36E-13	1.12E-14	2.24E-16
7	5.88E-12	1.34E-13	1.21E-14	2.76E-16
8	6.39E-12	1.34E-13	5.95E-15	1.25E-16
9	6.61E-12	1.33E-13	9.99E-16	2.01E-17
10	7.32E-12	1.33E-13	1.60E-14	2.91E-16
11	6.64E-12	1.33E-13	1.55E-14	3.10E-16
12	7.54E-12	1.41E-13	3.66E-15	6.82E-17

The use of the Excel matrix inversion function is about an order of magnitude less accurate than the double precision VBA code used in this assignment; that code uses standard Gaussian elimination with row pivoting only. The MATLAB code is 65 times more accurate than that VBA code. The MATLAB routine uses the QR method for solving problems that do not have special forms for their **A** matrices. It presumably also uses column pivoting