College of Engineering and Computer Science
Mechanical Engineering Department

Mechanical Engineering 501A
Seminar in Engineering Analysis

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September 20 Homework Solutions

1 – Find the spectrum (all the eigenvalues) and eigenvectors for the matrix, A**, at the right.**

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

We use the basic equation that the eigenvalues, λ, of a matrix, **A**, can be found from the equation Det(**A** – **I**λ) = **0**. Writing **A** – **I**λ and using the expression for the 3 by 3 determinant gives

$$Det(A - I\lambda) = \begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \begin{matrix} (3-\lambda)(4-\lambda)(1-\lambda) + (0)(0)(3) \\ +(0)(5)(6) - (0)(4-\lambda)(3) \\ -(0)(5)(1-\lambda) - (3-\lambda)(0)(6) \end{matrix} = (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

Here we see a case of two general results: (1) the determinant of a triangular matrix is the product of the terms on the diagonal; and (2) the eigenvalues of a triangular matrix are equal to the elements on the diagonal. In this case we have three unique eigenvalues, $\lambda_1 = 3$, $\lambda_2 = 4$, and $\lambda_3 = 1$. (Note that the ordering of the eigenvalues is arbitrary; often they are numbered in rank order. Regardless of how the eigenvalues are numbered, the eigenvectors must be numbered in a consistent manner.)

In solving for the eigenvectors, we recognize that it will not be possible to obtain a unique solution. The eigenvectors are determined only to within a multiplicative constant. If we denote the components of each eigenvector as x_1, x_2, and x_3, the solution for each eigenvector is found by substituting the corresponding eigenvalues into the equation (**A** – **I**λ)**x** = **0**, and solving for the corresponding x_i as the components of the eigenvector. This is shown for each eigenvector in the table below.

$5x_2 + 3x_3 = 0$ $x_2 + 6x_3 = 0$ $-2x_3 = 0$	For $\lambda_1 = 3$, we see that the solution to the third equation is $x_3 = 0$. With this solution, the second equation gives $x_2 = -6x_3 = 0$. We are left with no equation for x_1, so we conclude that this can be any value and we denote it is a. Thus gives the first eigenvector as $\mathbf{x}_{(1)} = [a\ \ 0\ \ 0]^T$.
$-x_1 + 5x_2 + 3x_3 = 0$ $6x_3 = 0$ $-2x_3 = 0$	For $\lambda_2 = 4$, the second and third equation both give $x_3 = 0$. The first equation gives $x_1 = 5x_2 + 3x_3 = 5x_2$. If we pick x_2 to be an arbitrary quantity, say b, then $x_1 = 5b$ and the second eigenvector becomes $\mathbf{x}_{(2)} = [5b\ \ b\ \ 0]^T$.
$2x_1 + 5x_2 + 3x_3 = 0$ $3x_2 + 6x_3 = 0$ $0 = 0$	For $\lambda_3 = 1$, ere the third equation gives us no information, but the second equation tells us that $3x_2 = -6x_3$. If we pick $x_3 = c$ (an arbitrary quantity), then $x_2 = 2c$ and the first equation gives $x_1 = (-5x_2 – 3x_3)/2 = (-5(-2c) – 3c)/2 = -7c/2$. So we have the third eigenvector as $\mathbf{x}_{(3)} = [7c/2\ \ -2c\ \ c]^T$.

2 – Find the principal axes and the corresponding expansion (or compression) factors for the elastic deformation matrix, A**, shown at the right.**

$$\mathbf{A} = \begin{bmatrix} 3/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{bmatrix}$$

As shown in the text, this matrix describes a stretching where every old coordinate, **x**, moves to a new coordinate, **y**., such that **y** = **Ax**.

The principal directions for this deformation are the directions for which the position vector changes only in magnitude. Thus, the new position vector, **y**, has the same direction as the old position vector, **x**. This means that the two vectors are related by a multiplicative constant. *E. g.*, **y** = λ**x**. This means that we have two equations: (1) the general equation **y** = **Ax**, for any coordinate direction and (2) **y** = λ**x**, in the principal coordinate direction. We can only satisfy these two equations if **Ax** = λ**x** for the principal coordinates. This gives an eigenvalues/eigenvector problem. In addition, the eigenvalue defines the factor by which the coordinate is expanded or contracted when the coordinates are multiplied by the eigenvalue, λ. Thus, the eigenvalue is the expansion or compression factor.

We first solve Det(**A** – **I**λ) = 0, as shown below, to get the eigenvalues.

$$Det(A - I\lambda) = \begin{vmatrix} 3/2 - \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & 1-\lambda \end{vmatrix} = \left(\frac{3}{2} - \lambda\right)(1-\lambda) - \left(\frac{1}{\sqrt{2}}\right)^2 = \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

The eigenvalues are found by the usual formula for the solution of a quadratic equation.

$$\lambda = \frac{\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 4(1)(1)}}{2(1)} = \frac{\frac{5}{2} \pm \sqrt{\left(\frac{25-16}{4}\right)}}{2(1)} = \frac{5 \pm 3}{4} \qquad \Rightarrow \qquad \lambda_1 = 2, \quad \lambda_2 = \frac{1}{2}$$

If we denote the components of each eigenvector as x_1, and x_2, the solution for each eigenvector is found by substituting the corresponding eigenvalues into the equation (**A** – **I**λ)**x** = **0**, and solving for the corresponding x_i as the components of the eigenvector. Here we have only two equations to solve, and one of the unknowns will be arbitrary. For $\lambda_1 = 2$, the two equations are $-x_1/2 + x_2/\sqrt{2} = 0$ and $x_1/\sqrt{2} + x_2 = 0$. Both these equations give us $x_1 = \sqrt{2}x_2$. Thus, our first eigenvector is $\mathbf{x}_{(1)} = [\sqrt{2}\,a \quad a]^T$, where a is arbitrary. For $\lambda_2 = ½$, the two equations are $x_1 + x_2/\sqrt{2} = 0$ and $x_1/\sqrt{2} + x_2/2 = 0$. Both these equations give us $x_2 = -\sqrt{2}x_1$ so that we can write our second eigenvector as $\mathbf{x}_{(2)} = [b \quad -\sqrt{2}\,b]^T$.

If we regard the x_1 coordinate as lying along a horizontal axis and the x_2 coordinate as lying along a vertical axis, we can find the principal coordinate directions corresponding to the two eigenvalues as follows.

$$\theta_1 = \tan^{-1}\left(\frac{x_{(1)2}}{x_{(1)1}}\right) = \tan^{-1}\left(\frac{a}{\sqrt{2}a}\right) = 0.61548 = 35.26^o \quad \text{for } \lambda_1 = 2$$

$$\theta_2 = \tan^{-1}\left(\frac{x_{(2)2}}{x_{(2)1}}\right) = \tan^{-1}\left(\frac{-\sqrt{2}b}{b}\right) = -0.955317 = -54.74^o \quad \text{for } \lambda_2 = 1/2$$

Here we find that the first principal direction is 35.26° with an expansion by a factor of 2 (the value of the eigenvalue); the second principal direction is –54.74° with a contraction by a factor of ½. Note that the numbering of “first” and “second” are arbitrary. Also, we see that when we are interested in only the direction of a vector, having a common factor in all components does not affect the result.

3 – Is the matrix at the right symmetric, skew-symmetric or orthogonal? Find the eigenvalues of the matrix and show how this illustrates Theorems 1 or Theorem 5 in section 8.3.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The matrix is ***not skew-symmetric*** since it has nonzero components on its principal diagonal. For a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}^T$), all terms on the principal diagonal must be zero.

The matrix is almost symmetric, but $a_{23} = -\sin(\theta) \neq a_{32} = \sin(\theta)$. Thus the matrix is ***not symmetric***.

If the matrix is orthogonal, the inner product of each pair of rows must vanish. There are three such products (ignoring order). Since the first row has no elements in the second or third columns and the second and third rows have no elements in the first column, the inner product of the first row, with the second and third row is zero. The inner product of the second and third row is $(0)(0) + [\cos(\theta)][\sin(\theta)] + [-\sin(\theta)][\cos(\theta)] = 0$. Since all possible inner products are zero we conclude that ***this matrix is orthogonal***.

We find the eigenvalues in the usual way by solving the equation Det($\mathbf{A} - \mathbf{I}\lambda$) = 0.

$$Det(A - I\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos(\theta)-\lambda & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta)-\lambda \end{vmatrix} = \begin{matrix} (1-\lambda)[\cos(\theta)-\lambda]^2 + (0)[\sin(\theta)](0) \\ +(0)(0)[-\sin(\theta)] - (0)[\cos(\theta)-\lambda](0) \\ -(0)(0)[\cos(\theta)-\lambda] - (1-\lambda)[\sin(\theta)][-\sin(\theta)] \end{matrix} = 0$$

The nonzero terms in the determinant result in the following equation.

$$\begin{aligned} &Det(A - I\lambda) = (1-\lambda)[\cos(\theta)-\lambda]^2 + (1-\lambda)[\sin^2(\theta)] \\ &= (1-\lambda)[\cos^2(\theta) - 2\lambda\cos(\theta) + \lambda^2 + \sin^2(\theta)] \\ &= (1-\lambda)[1 - 2\lambda\cos(\theta) + \lambda^2] = 0 \end{aligned}$$

We see that one root of this equation gives the eigenvalue $\lambda = 1$. We find the other two eigenvalues by setting the remaining factor in the final row equal to zero. This requires the solution of the following equation for the roots of the quadratic equation.

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4(1)(1)}}{2} = \cos(\theta) \pm \sqrt{[-\sin^2(\theta)]} = \cos(\theta) \pm i\sin(\theta)$$

Here $i^2 = -1$ indicating that we have a pair of complex eigenvalues. (This illustrates the general fact that real matrices can have complex eigenvalues, but such eigenvalues occur as pairs of complex numbers that are conjugate to each other.) The magnitude of each complex eigenvalue is equal to $\sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$. We see that the eigenvalues of this orthogonal matrix satisfy Theorem 5 on page 337, which states that the eigenvalues of an orthogonal matrix are real or complex conjugate pairs and have a magnitude of one.

4 – Give a geometrical interpretation of the coordinate transformation y **=** Ax**, with the matrix A as shown on the right.**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The individual components of the transformation **y** = **Ax**, with this matrix are shown below.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2\cos(\theta) - x_3\sin(\theta) \\ x_2\sin(\theta) + x_3\cos(\theta) \end{bmatrix}$$

This transformation from the old (x) coordinate system to the new (y) coordinate system represents a rotation in the plane perpendicular to the x_1 axis. This coordinate remains the same in the new system as indicated by the first equation $y_1 = x_1$. The remaining two equations, $y_2 = x_2\cos(\theta) - x_3\sin(\theta)$ and $y_3 = x_2\cos(\theta) + x_3\sin(\theta)$ give the location of the same point in the new coordinate system. These equations represent a counterclockwise rotation of the coordinate system through an angle θ.

5 – Find a basis for eigenvectors and diagonalize the matrix, A**, shown at the right.**

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

We start by solving Det(**A** – **I**λ) = 0, as shown below, to get the eigenvalues.

$$Det(A - I\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - (2)(1) = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

We see that the roots of this equation are 0 and 3 so that the two eigenvalues (giving the higher eigenvalue the lower index) are $\lambda_1 = 3$ and $\lambda_2 = 0$. If x_1 and x_2 are the two components of each eigenvector, we find these components by solving the equation (**A** – **I**λ) = **0**. For $\lambda_1 = 3$, we have to solve the equations $-x_1 + x_2 = 0$ and $2x_1 - 2x_2 = 0$. Both of these equations give $x_1 = x_2$. We can pick $x_1 = x_2 = 1$ for simplicity and write our first eigenvector as $\mathbf{x}_{(1)} = [1 \quad 1]^T$.

For $\lambda_2 = 0$, the equation (**A** – **I**λ) = **0**.gives $2x_1 + x_2 = 0$ for each equation. Thus, $x_2 = -2x_1$ for the second eigenvector. If we choose $x_1 = 1$ so that $x_2 = -2$, our second eigenvector is $\mathbf{x}_{(2)} = [1 \quad -2]^T$.

We have the general result that the matrix, **X**, whose columns are eigenvectors of the matrix **A** can be used to create a diagonal matrix, Λ = $\mathbf{X}^{-1}\mathbf{AX}$, provided that **X** has an inverse. The elements of the diagonal matrix are the eigenvalues. Thus in this case we expect that the matrix Λ will have [3 0] in its first row and [0 0] in its second row.

The **X** matrix in this case will be $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, and $\mathbf{X}^{-1}$ = $-\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$ from the equation for such an inverse in the lecture notes and in the Kreyszig text. You can readily verify that $\mathbf{XX}^{-1} = \mathbf{I}$. (Since our eigenvectors could be multiplied by any nonzero factor and still be eigenvectors, we can write the most general form of the **X** matrix in terms of arbitrary numbers a and b as **X** = $\begin{bmatrix} a & b \\ a & -2b \end{bmatrix}$. In this case, $\mathbf{X}^{-1}$ = $-\frac{1}{3ab}\begin{bmatrix} -2b & -b \\ -a & a \end{bmatrix}$. We still have $\mathbf{XX}^{-1} = \mathbf{I}$ for any choice of a and b that are not zero.)

The following operations show that $\mathbf{X}^{-1}\mathbf{AX}$ is a diagonal matrix of eigenvalues in this case.

$$\mathbf{X}^{-1}\mathbf{AX} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \bullet$$

$$\begin{bmatrix} (2)(1)+(1)(1) & (2)(1)+(1)(-2) \\ (2)(1)+(1)(1) & (2)(1)+(1)(-2) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} (-2)(3)+(-1)(3) & (2)(0)+(-1)(0) \\ (-1)(3)+(1)(3) & (-1)(0)+(1)(0) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -9 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

6 – Find out what kind of conic section (or pair of straight lines) is represented by the equation $9x_1^2 - 6x_1x_2 + x_2^2 = 40$. Transform it to principal axes. Express $x^T = [x_1 \;\; x_2]$ in terms of the new coordinate vector $y^T = [y_1 \;\; y_2]$.

We can write this equation as follows in matrix form.

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 40$$

To determine the kind of figure that this equation represents, we find the principal axes by finding the eigenvalues of the **A** matrix.

$$Det(A - I\lambda) = \begin{vmatrix} 9-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (9-\lambda)(1-\lambda) - (-3)(-3) = \lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$

We see that the roots of this equation are 0 and 10 so that the two eigenvalues (with the higher eigenvalue having the lower index) are $\lambda_1 = 10$ and $\lambda_2 = 0$. This means that the standard equation that is written in terms of the eigenvalues is $10y_1^2 + 0y_2^2 = 10y_1^2 = 40$. This is a pair of straight lines at $y_1 = \pm 2$.

To find the principal axes we have to find the eigenvectors. If x_1 and x_2 are the two components of each eigenvector, we find these components by solving the equation $(\mathbf{A} - \mathbf{I}\lambda) = \mathbf{0}$. For $\lambda_1 = 10$, we have to solve the equations $-x_1 - 3x_2 = 0$ and $-3x_1 - 9x_2 = 0$. Both of these equations give $x_1 = -3x_2$. We can pick $x_2 = 1$ and $x_1 = -3$ for simplicity and write our first eigenvector as $\mathbf{x}_{(1)} = [-3 \;\; 1]^T$.

For $\lambda_2 = 0$, the equation $(\mathbf{A} - \mathbf{I}\lambda) = \mathbf{0}$.gives $9x_1 - 3x_2 = 0$ and $-3x_1 + x_2 = 0$ for each equation. Thus, $x_2 = 3x_1$ for the second eigenvector. If we choose $x_1 = 1$ so that $x_2 = 3$, our second eigenvector is $\mathbf{x}_{(2)} = [1 \;\; 3]^T$.

Following example 6, we use these two eigenvectors to form the **X** matrix that relates the new (**y**) and old (**x**) coordinate systems by the equation **x** = **Xy**. Before doing this, we normalize the eigenvectors so that the two-norm of each eigenvector is one. The magnitude of each eigenvector is $3^2 + 1^2 = 10$. Thus, if we divide each component by $\sqrt{10}$, the norm of each eigenvector will be one. Doing this and constructing the **X** matrix gives.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{X}\mathbf{y} = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can also obtain $\mathbf{X}^{-1}$ from the equation in the course notes and the Kreyszig text for the inverse of a two-by two matrix. Doing this gives the following relation between the original and transformed coordinate system.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{X}^{-1}\mathbf{x} = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

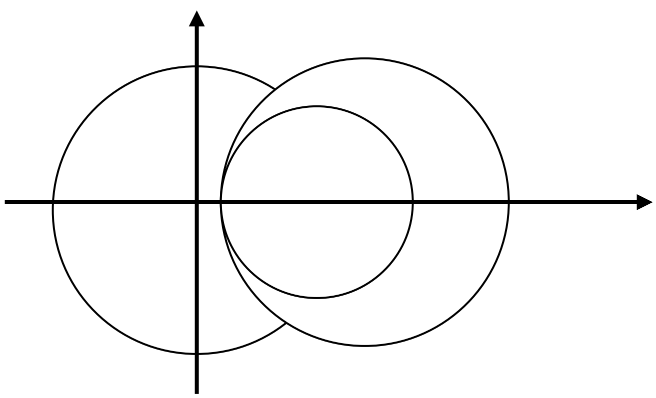
Here we see that $\mathbf{X} = \mathbf{X}^{-1}$. This is true because **X** is a symmetric, orthogonal matrix. (For an orthogonal matrix, $\mathbf{X}^{-1} = \mathbf{X}^T$. For a symmetric matrix, $\mathbf{X}^T = \mathbf{X}$. So, for a symmetric, orthogonal matrix, $\mathbf{X} = \mathbf{X}^{-1}$.

7 – Determine and sketch disks that contain the eigenvalues of the matrix A **at the right.**	$\mathbf{A} = \begin{bmatrix} 5 & -2 & 2 \\ 2 & 0 & 4 \\ 4 & 2 & 7 \end{bmatrix}$

Apply Gerschgorin's theorem which states that eigenvalues are located in a disk on the complex plane with a center given by the diagonal elements and a radius equal to the sum of the absolute values in the row, excluding the diagonal elements.

For the first row, the center is 5 and the radius is |-2|+|2| = 4. For the second row, the center is 0 and the radius is |2| + |4| = 6. For the third row, the center is 7 and the radius is |4|+|2| = 6.

The resulting Gerschgorin disks are shown on the next page. We see that the disks overlap on the real axis so that we cannot separate out any eigenvalues. In addition, **A** is not symmetric so we cannot be sure that the eigenvalues line on the real axis.



8 – Find the similarity transformation, $T^{-1}AT$ **such that the Gerschgorin disk with a center at 5 for the matrix** A **at the right is reduced to 1/100 of its original value.**	$\mathbf{A} = \begin{bmatrix} 5 & 10^{-2} & 10^{-2} \\ 10^{-2} & 8 & 10^{-2} \\ 10^{-2} & 10^{-2} & 9 \end{bmatrix}$

The required similarity transformation uses a diagonal matrix, **T**, where all terms on the diagonal are one, except the one in the row whose disk radius we seek to reduce. To see that this is the case, consider the basic form of **T** as a diagonal matrix with components $t_j\delta_{ij}$. The typical term in the inverse matrix, $\mathbf{T}^{-1}$ will be δ_{ij}/t_j. The components of the $\mathbf{T}^{-1}\mathbf{AT}$ product are found from the usual rules for matrix multiplication with the recognition that $\delta_{ij} = 0$ unless i = j. Thus the only nonzero terms in a summation in which δ_{ij} is a factor is the term for which the i = j.

$$[\mathbf{T}^{-1}\mathbf{AT}]_{ij} = \sum_{k=1}^{n}\left(\frac{\delta_{ik}}{t_k}\sum_{m=1}^{n} a_{km}t_m\delta_{mj}\right) = \sum_{k=1}^{n}\frac{\delta_{ik}}{t_k}a_{kj}t_j = \frac{t_j}{t_i}a_{ij}$$

In the transformation matrix we seek here, all the diagonal terms will be one, except for the chosen row, say row k, in which t_k will be a large factor. Thus there are three possible results from this transformation with the single diagonal factor, F, not equal to one.

$$[\mathbf{T}^{-1}\mathbf{AT}]_{ij} = \begin{cases} Fa_{ij} & i \neq j \text{ and factor } F \text{ located in row } j \\ \dfrac{a_{ij}}{F} & i \neq j \text{ and factor } F \text{ located in row } i \\ a_{ij} & i = j \text{ or factor } F \text{ not located in row } i \text{ or row } j \end{cases}$$

This leads to the basic rule for using a $\mathbf{T}^{-1}\mathbf{AT}$ transformation to reduce the size of Gerschorgin disk radii. To reduce the radius around the center from row k by a factor of 1/F, use a diagonal matrix with all ones on the diagonal except for a value of F in row k. Be careful though. This will increase the values of the elements in column k by a factor of F in the result.

In this case we want to decrease the disk radius by a factor of 1/100 for the disk with a center given by the first row. To do this we use a (diagonal) **T** matrix with a 100 in the first row and 1's for all other diagonal elements. This matrix and the result of the transformation is shown below.

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} .01 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 10^{-2} & 10^{-2} \\ 10^{-2} & 8 & 10^{-2} \\ 10^{-2} & 10^{-2} & 9 \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .01 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 500 & 10^{-2} & 10^{-2} \\ 1 & 8 & 10^{-2} \\ 1 & 10^{-2} & 9 \end{bmatrix} = \begin{bmatrix} 5 & 10^{-4} & 10^{-4} \\ 1 & 8 & 10^{-2} \\ 1 & 10^{-2} & 9 \end{bmatrix}$$

We see that this transformation has decreased the disk radius for the first row by a factor of 100 as desired. However, it has also increased the disk radius for the second and third rows from 0.02 in the original matrix to 1.02 in the transformed matrix.

This approach to reducing the disk radius is based on the fact that the eigenvalues of $\mathbf{T}^{-1}\mathbf{AT}$, for any invertible matrix, **T**, are the same as the eigenvalues of **A**. We can easily show this since the eigenvalues of $\mathbf{T}^{-1}\mathbf{AT}$ are defined by the equation $\mathbf{T}^{-1}\mathbf{ATx} = \lambda\mathbf{x}$. If we premultiply this equation by **T**, we get the result that $\mathbf{TT}^{-1}\mathbf{ATx} = \mathbf{IATx} = \mathbf{ATx} = \lambda\mathbf{Tx}$. This final equation shows that the eigenvalues of **A** and $\mathbf{T}^{-1}\mathbf{AT}$ are the same. (The eigenvectors are different, however. The eigenvectors of $\mathbf{T}^{-1}\mathbf{AT}$, are **x**, the eigenvectors of **A** are **Tx**.)